

NEW NON-NATURALLY REDUCTIVE EINSTEIN METRICS ON EXCEPTIONAL SIMPLE LIE GROUPS

HUIBIN CHEN, ZHIQI CHEN AND SHAOQIANG DENG

ABSTRACT. In this article, we achieved several non-naturally reductive Einstein metrics on exceptional simple Lie groups, which are formed by the decomposition arising from general Wallach spaces. By using the decomposition corresponding to the two involutive automorphisms, we calculated the non-zero coefficients in the expression for the components of Ricci tensor with respect to the given metrics. The Einstein metrics are obtained as solutions of systems polynomial equations, which we manipulate by symbolic computations using Gröbner bases.

1. INTRODUCTION

A Riemannian manifold (M, g) is called Einstein if there exists a constant $\lambda \in \mathbb{R}$ such that the Ricci tensor r with respect to g satisfies $r = \lambda g$. The readers can go to Besse's book [7] for more details and results in this field before 1986. General existence results are difficult to obtain, as a result, many mathematicians pay more attention on the special examples for Einstein manifolds. Among the first important attempts, the works of G. Jensen [18] and M. Wang, W. Ziller [25] made much contributions to the progress of this field. When the problem is restricted to Lie groups, D'Atri and Ziller in [14] obtained a large number of naturally reductive left-invariant metrics when G is simple. Also in this paper, they raised a problem: Whether there exists non-naturally reductive Einstein metrics on compact Lie group G ?

In 1994, K. Mori [19] discovered the first left-invariant Einstein metrics on compact simple Lie groups $SU(n)$ for $n \geq 6$, which are non-naturally reductive. In 2008, Arvanitoyeorgos, Mori and Sakane proved the existence of new non-naturally reductive Einstein metrics for $SO(n)(n \geq 11)$, $Sp(n)(n \geq 3)$, E_6 , E_7 and E_8 , using fibrations of a compact simple Lie group over a Kähler C -space with two isotropy summands (see [2]). In 2014, by using the methods of representation theory, Chen and Liang [11] found a non-naturally reductive Einstein metric on the compact simple Lie group F_4 . More recently, Chrysikos and Sakane proved that there exists non-naturally reductive Einstein metric on exceptional Lie groups, especially for G_2 , they gave the first example of non-naturally reductive Einstein metric (see [13]).

In this paper we consider new non-naturally reductive Einstein metrics on compact exceptional Lie groups G which can be seen as the principal bundle over generalized Wallach spaces $M = G/K$. In 2014, classification for generalized Wallach spaces arising from a compact simple Lie group has been obtained

by Nikonorov [22] and Chen, Kang and Liang [12], in particular, Nikonorov investigated the semi-simple case and gave the classification in [22].

As is known to all, the involutive automorphisms play an important role in the development of homogeneous geometry. The Riemannian symmetric pairs were classified by Cartan [10], in Lie algebra level, which can be treated as the structure of a Lie algebra with an involutive automorphism satisfying some topologic properties. Later on, the more general semi-simple symmetric pairs were studied by Berger [6], whose classification can be obtained in the view of involutive automorphism. Recently, Huang and Yu [16] classified the Klein four subgroups Γ of $\text{Aut}(\mathfrak{u}_0)$ for each compact Lie algebra \mathfrak{u}_0 up to conjugation by calculating the symmetric subgroups $\text{Aut}(\mathfrak{u}_0)^\theta$ and their involution classes.

According to the article [12], each kind of generalized Wallach spaces arising from simple Lie groups is associated with two commutative involutive automorphisms of \mathfrak{g} , the Lie algebra of G . With these two involutive automorphisms, we have two different corresponding decompositions of \mathfrak{g} , which are in fact irreducible symmetric pairs. According to these two irreducible symmetric pairs, we can get some linear equations for the non-zero coefficients in the expression of components of Ricci tensor with respect to the given metric. With the help of computer, we get the Einstein metrics from the solutions of systems polynomial equations. We mainly deal with two kinds of generalized Wallach spaces, one of which is without centers in \mathfrak{k} and the other is with a 1-dimensional center in \mathfrak{k} . Along with the results in [11], we list all the number of non-naturally reductive left-invariant Einstein metrics on exceptional simple Lie groups G arising from generalized Wallach spaces with no center in \mathfrak{k} . In this table, we still use the

G	Types	K	$p + q$	N_{non-nn}
F_4	$F_4\text{-I}$	$\text{SO}(8)$	$1 + 3$	1[11]
	$F_4\text{-II}$	$\text{SU}(2) \times \text{SU}(2) \times \text{SO}(5)$	$3 + 3$	3 new
E_6	$E_6\text{-III}$	$\text{SU}(2) \times \text{Sp}(3)$	$2 + 3$	4 new
	$E_6\text{-II}$	$\text{U}(1) \times \text{SU}(2) \times \text{SU}(2) \times \text{SU}(4)$	$0 + 3 + 3$	7 new
E_7	$E_7\text{-I}$	$\text{SU}(2) \times \text{SU}(2) \times \text{SU}(2) \times \text{SO}(8)$	$4 + 3$	7 new
	$E_7\text{-III}$	$\text{SO}(8)$	$1 + 3$	1
	$E_7\text{-II}$	$\text{U}(1) \times \text{SU}(2) \times \text{SU}(6)$	$0 + 2 + 3$	6 new
E_8	$E_8\text{-I}$	$\text{SU}(2) \times \text{SU}(2) \times \text{SO}(12)$	$3 + 3$	11 new
	$E_8\text{-II}$	$\text{Ad}(\text{SO}(8) \times \text{SO}(8))$	$2 + 3$	2 new

TABLE 1. Number of non-naturally reductive left-invariant Einstein metrics on exceptional simple Lie group G arising from generalized Wallach spaces

notations in [12] to represent the type of generalized Wallach space, N_{non-nn} represents the number of non-naturally reductive Einstein metrics on G and p, q coincides with the indices in the decomposition $\mathfrak{g} = \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_p \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q$, in fact $q = 3$ for all types.

We describe our results in the following theorem.

- Theorem 1.1.** (1) *The compact simple Lie group F_4 admits at least 6 new non-naturally reductive and non-isometric left-invariant Einstein metrics. These metrics are $\text{Ad}(\text{SU}(2) \times \text{SU}(2) \times \text{SO}(5))$ -invariant.*
- (2) *The compact simple Lie group E_6 admits at least 11 new non-naturally reductive and non-isometric left-invariant Einstein metrics. Four of these metrics are $\text{Ad}(\text{SU}(2) \times \text{Sp}(3))$ -invariant and the other 7 are $\text{Ad}(\text{U}(1) \times \text{SU}(2) \times \text{SU}(2) \times \text{SU}(4))$ -invariant.*
- (3) *The compact simple Lie group E_7 admits at least 13 new non-naturally reductive and non-isometric left-invariant Einstein metrics. Seven of these metrics are $\text{Ad}(\text{SU}(2) \times \text{SU}(2) \times \text{SU}(2) \times \text{SO}(8))$ -invariant and the other 6 are $\text{Ad}(\text{U}(1) \times \text{SU}(2) \times \text{SU}(6))$ -invariant.*
- (4) *The compact simple Lie group E_8 admits at least 13 new non-naturally reductive and non-isometric left-invariant Einstein metrics. Two of these metrics are $\text{Ad}(\text{SO}(8) \times \text{SO}(8))$ -invariant and the other 11 are $\text{Ad}(\text{SU}(2) \times \text{SU}(2) \times \text{SO}(12))$ -invariant.*

The paper is organized as follows: In section 2 we will recall a formula for the Ricci tensor of G when we see G as a homogeneous space. In section 3 we will introduce how we operate our methods to solve the non-zero coefficients in the expressions for Ricci tensor, where we will classify the exceptional Lie groups by the number of simple ideals of \mathfrak{k} . Then for each case in section 3, we will discuss the non-naturally reductive Einstein metrics via the solutions of systems polynomial equations, which will be described in section 4.

2. THE RICCI TENSOR FOR REDUCTIVE HOMOGENEOUS SPACES

In this section we will recall an expression for the Ricci tensor with respect to a class of given metrics on a compact semi-simple Lie group and figure out whether a metric on G is naturally reductive.

Let G be a compact semi-simple Lie group with Lie algebra \mathfrak{g} , K a connected closed subgroup of G with Lie algebra \mathfrak{k} . Through this paper, we denote by B the negative of the Killing form of \mathfrak{g} , which is positive definite because of the compactness of G , as a result, B can be treated as an inner product on \mathfrak{g} . Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the reductive decomposition with respect to B such that $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$, where \mathfrak{m} is the tangent space of G/K . We assume that \mathfrak{m} can be decomposed into mutually non-equivalent irreducible $\text{Ad}(K)$ -modules as follows:

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q.$$

We will write $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_p$, where $\mathfrak{k}_0 = Z(\mathfrak{k})$ is the center of \mathfrak{k} and \mathfrak{k}_i is the simple ideal for $i = 1, \dots, p$. Let $G \times K$ act on G by $(g_1, g_2)g = g_1 g g_2^{-1}$, then $G \times K$ acts almost effectively on G with isotropy group $\Delta(K) = \{(k, k) | k \in K\}$. As a result, G can be treated as the coset space $(G \times K)/\Delta(K)$

and we have $\mathfrak{g} \oplus \mathfrak{k} = \Delta(\mathfrak{k}) \oplus \Omega$, where $\Omega \cong T_0((G \times K)/\Delta(K)) \cong \mathfrak{g}$ via the linear map $(X, Y) \rightarrow (X - Y) \in \mathfrak{g}$, $(X, Y) \in \Omega$.

As is known, there exists an 1-1 corresponding between all G -invariant metrics on the reductive homogeneous space G/K and $\text{Ad}_G(K)$ -invariant inner products on \mathfrak{m} . A Riemannian homogeneous space $(M = G/K, g)$ with reductive complement \mathfrak{m} of \mathfrak{k} in \mathfrak{g} is called *naturally reductive* if

$$([X, Y]_{\mathfrak{m}}, Z) + (Y, [X, Z]_{\mathfrak{m}}) = 0,$$

where $X, Y, Z \in \mathfrak{m}$, (\cdot, \cdot) is the corresponding inner product on \mathfrak{g} .

In [14], D'Atri and Ziller study the naturally reductive metrics among left invariant metrics on compact Lie groups and they obtained a complete classification of such metrics in the simple case. The following theorem will play an important role to decide whether a left-invariant metric on a Lie group is naturally reductive.

Theorem 2.1. *For any inner product b on the center \mathfrak{k}_0 of \mathfrak{k} , the following left-invariant metrics on G is naturally reductive with respect to the action $(g, k)y = gyk^{-1}$ of $G \times K$:*

$$\langle \cdot, \cdot \rangle = u_0 b|_{\mathfrak{k}_0} + u_1 B|_{\mathfrak{k}_1} + \cdots + u_p B|_{\mathfrak{k}_p} + x B|_{\mathfrak{m}}, \quad (u_0, u_1, \dots, u_p, x \in \mathbb{R}^+)$$

Conversely, if a left-invariant metric $\langle \cdot, \cdot \rangle$ on a compact simple Lie group G is naturally reductive, then there exists a closed subgroup K of G such that $\langle \cdot, \cdot \rangle$ can be written as above.

Now we have a orthogonal decomposition of \mathfrak{g} with respect to the Killing form of \mathfrak{g} : $\mathfrak{g} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_p \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q = (\mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_p) \oplus (\mathfrak{k}_{p+1} \oplus \cdots \oplus \mathfrak{k}_{p+q})$, with $\mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q = \mathfrak{k}_{p+1} \oplus \cdots \oplus \mathfrak{k}_{p+q}$, respectively. In addition, we assume that $\dim_{\mathbb{R}} \mathfrak{k}_0 \leq 1$ and the ideals \mathfrak{k}_i are mutually non-isomorphic for $i = 1, \dots, p$. Then we consider the following left-invariant metric on G which is in fact $\text{Ad}(K)$ -invariant:

$$\langle \cdot, \cdot \rangle = x_0 \cdot B|_{\mathfrak{k}_0} + x_1 \cdot B|_{\mathfrak{k}_1} + \cdots + x_{p+q} \cdot B|_{\mathfrak{k}_{p+q}}, \quad (2.1)$$

where $x_i \in \mathbb{R}^+$ for $i = 1, \dots, p+q$.

Set from now on $d_i = \dim_{\mathbb{R}} \mathfrak{k}_i$ and $\{e_{\alpha}^i\}_{\alpha=1}^{d_i}$ be a B -orthonormal basis adapted to the decomposition of \mathfrak{g} which means $e_{\alpha}^i \in \mathfrak{k}_i$ and α is the number of basis in \mathfrak{k}_i . Then we consider the numbers $A_{\alpha, \beta}^{\gamma} = B([e_{\alpha}^i, e_{\beta}^j], e_{\gamma}^k)$ such that $[e_{\alpha}^i, e_{\beta}^j] = \sum_{\gamma} A_{\alpha, \beta}^{\gamma} e_{\gamma}^k$, and set

$$(ijk) := \left[\begin{smallmatrix} i \\ j \ k \end{smallmatrix} \right] = \sum (A_{\alpha, \beta}^{\gamma})^2,$$

where the sum is taken over all indices α, β, γ with $e_{\alpha}^i \in \mathfrak{k}_i, e_{\beta}^j \in \mathfrak{k}_j, e_{\gamma}^k \in \mathfrak{k}_k$. Then (ijk) is independent of the choice for the B -orthonormal basis of $\mathfrak{k}_i, \mathfrak{k}_j, \mathfrak{k}_k$, and symmetric for all three indices which means $(ijk) = (jik) = (jki)$.

In [2] and [23], the authors obtained the formulas for the components of Ricci tensor with respect to the left-invariant metric given by (2.1), which can be described by the following lemma:

Lemma 2.2. *Let G be a compact connected semi-simple Lie group endowed with the left-invariant metric $\langle \cdot, \cdot \rangle$ given by (2.1). Then the components r_0, r_1, \dots, r_{p+q} of the Ricci tensor associated to $\langle \cdot, \cdot \rangle$ are expressed as follows:*

$$r_k = \frac{1}{2x_k} + \frac{1}{4d_k} \sum_{j,i} \frac{x_k}{x_j x_i} \begin{bmatrix} k \\ j \ i \end{bmatrix} - \frac{1}{2d_k} \sum_{j,i} \frac{x_j}{x_k x_i} \begin{bmatrix} j \\ k \ i \end{bmatrix}, \quad (k = 0, 1, \dots, p+q).$$

Here, the sums are taken over all $i = 0, 1, \dots, p+q$. In particular, for each k it holds that

$$\sum_{i,j}^{p+q} \begin{bmatrix} j \\ k \ i \end{bmatrix} = \sum_{ij}^{p+q} (kij) = d_k.$$

3. CALCULATIONS FOR NON-ZERO COEFFICIENTS IN THE EXPRESSIONS OF RICCI TENSOR

In this section, we will calculate the non-zero coefficients in the expressions for the components of the Ricci tensor with respect to the given metric (2.1). First of all, we classify the exceptional Lie groups without center in K into the following three types, namely $p = 2$, $p = 3$ and $p = 4$, where p represents the number of the simple ideals of \mathfrak{k} . For the case of $p = 1$, according to the classification [12], there are only F_4 -I and E_7 -III, the non-naturally reductive Einstein metrics on which were studied in [11] and [Lei], respectively.

We recall the definition of generalized Wallach spaces. Let G/K be a reductive homogeneous space, where G is a semi-simple compact connected Lie group, K is a connected closed subgroup of G , \mathfrak{g} and \mathfrak{k} are the corresponding Lie algebras, respectively. If \mathfrak{m} , the tangent space of G/K at $o = \pi(e)$, can be decomposed into three $\text{ad}(\mathfrak{k})$ -invariant irreducible summands pairwise orthogonal with respect to B as:

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3,$$

satisfying $[\mathfrak{m}_i, \mathfrak{m}_i] \in \mathfrak{k}$ for $i \in \{1, 2, 3\}$ and $[\mathfrak{m}_i, \mathfrak{m}_j] \in \mathfrak{m}_k$ for $\{i, j, k\} = \{1, 2, 3\}$, then we call G/K a generalized Wallach space.

In [12] and [22], the authors gave the complete classification of generalized Wallach spaces in the simple case. Here we use the notations in [12], then for $p = 2$, there are E_6 -III and E_8 -II, for $p = 3$, there are F_4 -II and E_8 -I and for $p = 4$ there is only E_7 -I. For later use, we introduce the following lemma given in [1].

Lemma 3.1. *Let $\mathfrak{q} \subset \mathfrak{r}$ be arbitrary subalgebra in \mathfrak{g} with \mathfrak{q} simple. Consider in \mathfrak{q} an orthonormal (with respect to $B_{\mathfrak{r}}$) basis $\{f_j\} (1 \leq j \leq \dim(\mathfrak{q}))$. Then*

$$\sum_{i,j,k=1}^{\dim(\mathfrak{q})} (B_{\mathfrak{r}}([f_i, f_j], f_k))^2 = \alpha_{\mathfrak{q}}^{\mathfrak{r}} \cdot \dim(\mathfrak{q}),$$

where α_q^r is determined by the equation $B_q = \alpha_q^r \cdot B_r|_q$.

For $p = 2$, we consider the following metrics on \mathfrak{g} :

$$\langle , \rangle = x_1 B|_{\mathfrak{k}_1} + x_2 B|_{\mathfrak{k}_2} + x_3 B|_{\mathfrak{m}_1} + x_4 B|_{\mathfrak{m}_2} + x_5 B|_{\mathfrak{m}_3}. \quad (3.1)$$

Since $\mathfrak{k}_1, \mathfrak{k}_2$ are the simple ideals of \mathfrak{k} and by the definition of generalized Wallach space, we can easily obtain the possible non-zero coefficients in the expressions for Ricci tensor as follows:

$$(111), (222), (133), (144), (155), (233), (244), (255), (345)$$

According to Lemma 2.2, we have

$$\begin{aligned} r_1 &= \frac{1}{4d_1} \left(\frac{1}{x_1} (111) + \frac{x_1}{x_3^2} (133) + \frac{x_1}{x_4^2} (144) + \frac{x_1}{x_5^2} (155) \right), \\ r_2 &= \frac{1}{4d_2} \left(\frac{1}{x_2} (222) + \frac{x_2}{x_3^2} (233) + \frac{x_2}{x_4^2} (244) + \frac{x_2}{x_5^2} (255) \right), \\ r_3 &= \frac{1}{2x_3} + \frac{1}{2d_3} (345) \left(\frac{x_3}{x_4 x_5} - \frac{x_4}{x_3 x_5} - \frac{x_5}{x_3 x_4} \right) - \frac{1}{2d_3} \left(\frac{x_1}{x_3^2} (133) + \frac{x_2}{x_3^2} (233) \right), \\ r_4 &= \frac{1}{2x_4} + \frac{1}{2d_4} (345) \left(\frac{x_4}{x_3 x_5} - \frac{x_3}{x_4 x_5} - \frac{x_5}{x_3 x_4} \right) - \frac{1}{2d_4} \left(\frac{x_1}{x_4^2} (144) + \frac{x_2}{x_4^2} (244) \right), \\ r_5 &= \frac{1}{2x_5} + \frac{1}{2d_5} (345) \left(\frac{x_5}{x_4 x_3} - \frac{x_4}{x_3 x_5} - \frac{x_3}{x_5 x_4} \right) - \frac{1}{2d_5} \left(\frac{x_1}{x_5^2} (155) + \frac{x_2}{x_5^2} (255) \right), \end{aligned}$$

and

$$\begin{aligned} (111) + (133) + (144) + (155) &= d_1, \\ (222) + (233) + (244) + (255) &= d_2, \\ 2(133) + 2(233) + 2(345) &= d_3, \\ 2(144) + 2(244) + 2(345) &= d_4, \\ 2(155) + 2(255) + 2(345) &= d_5. \end{aligned} \quad (3.2)$$

We used the symmetric property of three indices in (ijk) in above equations.

In order to calculate the non-zero coefficients $(111), (222), (133), (144), (155), (233), (244), (255), (345)$, we should know more about the structure of the corresponding Lie algebras. Since the structure of generalized Wallach space arising from a simple group can be decided by two commutative involutive automorphisms on \mathfrak{g} , we can learn more information from these two automorphisms.

Lemma 3.2. *In the case of $p = 2$, the non-zero coefficients in the components of Ricci tensor with respect to metric (3.1) are as follows:*

for $E_6\text{-III}$, the non-zero coefficients are

$$\begin{aligned} (111) &= \frac{1}{2}, (133) = 0, (144) = \frac{7}{4}, (155) = \frac{3}{4}, \\ (222) &= 7, (233) = \frac{7}{2}, (244) = \frac{35}{4}, (255) = \frac{7}{4}, (345) = \frac{7}{2}, \end{aligned}$$

for E_8-II , the non-zero coefficients are

$$\begin{aligned} (111) = (222) &= \frac{28}{5}, (345) = \frac{256}{15}, \\ (133) = (144) = (233) = (244) = (155) = (255) &= \frac{112}{15}. \end{aligned}$$

Proof. Case of E_6-III . For this case, we denote the two commutative involutive automorphisms by θ and τ . By the conclusions in [12], we know that each of the automorphisms corresponds to an irreducible symmetric pair. For the structure of E_6-III , we have the following decomposition:

$$\mathfrak{g} = A_1 \oplus C_3 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3.$$

Let

$$\mathfrak{g} = \mathfrak{b} + \mathfrak{p}, \quad \mathfrak{b} = \mathfrak{b}_1 + \mathfrak{b}_2, \quad \mathfrak{b}_1 = A_1, \quad \mathfrak{b}_2 = C_3 + \mathfrak{m}_1 \cong A_5, \quad \mathfrak{p} = \mathfrak{m}_2 + \mathfrak{m}_3, \quad (3.3)$$

then $(\mathfrak{g}, \mathfrak{b})$ is in fact the irreducible symmetric pair corresponding to θ .

Let

$$\mathfrak{g} = \mathfrak{b}' + \mathfrak{p}', \quad \mathfrak{b}' = A_1 + C_3 + \mathfrak{m}_2 \cong F_4, \quad \mathfrak{p}' = \mathfrak{m}_1 + \mathfrak{m}_3, \quad (3.4)$$

then $(\mathfrak{g}, \mathfrak{b}')$ is in fact the irreducible symmetric pair corresponding to τ . We consider the following metric on \mathfrak{g} with respect to the decomposition (3.3)

$$<<, >>_1 = u_1 B|_{A_1} + u_2 B|_{A_5} + u_3|_{\mathfrak{p}}, \quad (3.5)$$

and denote the components of the Ricci tensor with respect to the metric $<<, >>_1$ by \tilde{r}_1, \tilde{r}_2 and \tilde{r}_3 . If we let $x_1 = u_1, x_2 = x_3 = u_2, x_4 = x_5 = u_3$, then the metric given by (3.1) and (3.5) are the same, thus their corresponding Ricci tensor are also the same, which means $r_1 = \tilde{r}_1, r_2 = r_3 = \tilde{r}_2, r_4 = r_5 = \tilde{r}_3$. As a result, from the expressions of each Ricci components, we have

$$\begin{aligned} \frac{1}{4d_2} ((222) + (233)) &= \frac{1}{4} - \frac{1}{2d_3}(345), \\ \frac{1}{4d_2} ((244) + (255)) &= \frac{1}{2d_3}(345), \\ \frac{1}{2d_4}(144) &= \frac{1}{2d_5}(155), \\ (133) &= 0. \end{aligned} \quad (3.6)$$

With the same method, we consider the irreducible symmetric pair corresponding to the involutive automorphism τ . The metric taken into consideration is as follows:

$$<<, >>_2 = w_1 B|_{F_4} + w_2 B|_{\mathfrak{p}'}, \quad (3.7)$$

If we let $x_1 = x_2 = x_4 = w_1$ and $x_3 = x_5 = w_2$, the Ricci tensors with respect to (3.1) and (3.7) are the same, by comparing the expression for both components, we have

$$\begin{aligned} \frac{1}{4d_1} ((111) + (144)) &= \frac{1}{4d_2} ((222) + (244)) = \frac{1}{4} - \frac{1}{2d_4}(345), \\ \frac{1}{4d_1} ((133) + (155)) &= \frac{1}{4d_2} ((233) + (255)) = \frac{1}{2d_4}(345). \end{aligned} \quad (3.8)$$

We can easily calculate $\alpha_{A_1}^{E_6} = \frac{8}{48}$, and $\alpha_{C_3}^{E_6} = \frac{16}{48}$, since their corresponding roots have the same length, we obtain $(111) = \frac{1}{2}$ and $(222) = 7$ according to Lemma 3.1. From Table 1 in [22], we can find out $(345) = \frac{7}{2}$, along with the equations (3.2), (3.6) and (3.8) and $d_1 = 3$, $d_2 = 21$, $d_3 = 14$, $d_4 = 28$ and $d_5 = 12$, one can easily get the solutions given in the lemma.

Case of E_8 -II. For this case, we have the following decomposition:

$$\mathfrak{g} = D_4 \oplus D_4 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3.$$

Then we study the two commutative involutive automorphisms denoted by θ and τ . In fact, the corresponding irreducible symmetric pair $(\mathfrak{g}, \mathfrak{b})$ and $(\mathfrak{g}, \mathfrak{b}')$ of θ and τ are respectively as follows:

$$\mathfrak{g} = \mathfrak{b} + \mathfrak{p}, \quad \mathfrak{b} = D_4 + D_4 + \mathfrak{m}_1 \cong D_8, \quad \mathfrak{p} = \mathfrak{m}_2 + \mathfrak{m}_3.$$

$$\mathfrak{g} = \mathfrak{b}' + \mathfrak{p}', \quad \mathfrak{b}' = D_4 + D_4 + \mathfrak{m}_2 \cong D_8, \quad \mathfrak{p} = \mathfrak{m}_1 + \mathfrak{m}_3.$$

By the same methods operated in the case of E_6 -III, we have

$$\begin{aligned} (111) &= (222) = \frac{28}{5}, (345) = \frac{256}{15}, \\ (133) &= (144) = (155) = (233) = (244) = (255) \frac{112}{15}. \end{aligned}$$

□

For $p = 3$, we have the following decomposition:

$$\mathfrak{g} = \mathfrak{k}_1 + \mathfrak{k}_2 + \mathfrak{k}_3 + \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3,$$

and we consider the metric as follows:

$$\langle \cdot, \cdot \rangle = x_1 B|_{\mathfrak{k}_1} + x_2 B|_{\mathfrak{k}_2} + x_3 B|_{\mathfrak{k}_3} + x_4 B|_{\mathfrak{p}_1} + x_5 B|_{\mathfrak{p}_2} + x_6 B|_{\mathfrak{p}_3}. \quad (3.9)$$

Since $\mathfrak{k}_i (i = 1, 2, 3)$ is simple ideal of \mathfrak{k} and according to the structure of generalized Wallach space, it is easy to know that the possible non-zero coefficients in the expression for the components of Ricci tensor with respect to the metric (3.9) are as follows:

$$(111), (222), (333), (144), (155), (166), (244), (255), (266), (344), (355), (366), (456).$$

Then the components of the Ricci tensor with respect to this metric are as follows according to the Lemma 2.2:

$$\begin{aligned}
r_1 &= \frac{1}{4d_1} \left(\frac{1}{x_1}(111) + \frac{x_1}{x_4^2}(144) + \frac{x_1}{x_5^2}(155) + \frac{x_1}{x_6^2}(166) \right), \\
r_2 &= \frac{1}{4d_2} \left(\frac{1}{x_2}(222) + \frac{x_2}{x_4^2}(244) + \frac{x_2}{x_5^2}(255) + \frac{x_2}{x_6^2}(266) \right), \\
r_3 &= \frac{1}{4d_3} \left(\frac{1}{x_3}(333) + \frac{x_3}{x_4^2}(344) + \frac{x_3}{x_5^2}(355) + \frac{x_3}{x_6^2}(366) \right), \\
r_4 &= \frac{1}{2x_4} + \frac{1}{2d_4}(456) \left(\frac{x_4}{x_5x_6} - \frac{x_6}{x_4x_5} - \frac{x_5}{x_4x_6} \right) - \frac{1}{2d_4} \left(\frac{x_1}{x_4^2}(144) + \frac{x_2}{x_4^2}(244) + \frac{x_3}{x_4^2}(344) \right), \\
r_5 &= \frac{1}{2x_5} + \frac{1}{2d_5}(456) \left(\frac{x_5}{x_4x_6} - \frac{x_6}{x_4x_5} - \frac{x_4}{x_5x_6} \right) - \frac{1}{2d_5} \left(\frac{x_1}{x_5^2}(155) + \frac{x_2}{x_5^2}(255) + \frac{x_3}{x_5^2}(355) \right), \\
r_6 &= \frac{1}{2x_6} + \frac{1}{2d_6}(456) \left(\frac{x_6}{x_5x_4} - \frac{x_4}{x_6x_5} - \frac{x_5}{x_4x_6} \right) - \frac{1}{2d_6} \left(\frac{x_1}{x_6^2}(166) + \frac{x_2}{x_4^2}(266) + \frac{x_3}{x_4^2}(366) \right).
\end{aligned}$$

and the following equations:

$$\begin{aligned}
(111) + (144) + (155) + (166) &= d_1, \\
(222) + (244) + (255) + (266) &= d_2, \\
(333) + (344) + (355) + (366) &= d_3, \\
2(144) + 2(244) + 2(344) + 2(456) &= d_4, \\
2(155) + 2(255) + 2(355) + 2(456) &= d_5, \\
2(166) + 2(266) + 2(366) + 2(456) &= d_6.
\end{aligned} \tag{3.10}$$

Lemma 3.3. *The possible non-zero coefficients in the expression for the components of Ricci tensor with respect to the metric (3.9) are as follows:*

for case of F_4 -II, we have

$$\begin{aligned}
(111) = (222) &= \frac{2}{3}, (333) = \frac{10}{3}, (144) = (244) = \frac{5}{3}, (155) = (266) = 0, \\
(166) = (255) &= \frac{2}{3}, (355) = (366) = \frac{10}{9}, (344) = \frac{40}{9}, (456) = \frac{20}{9}.
\end{aligned}$$

for case of E_8 -I, we have

$$\begin{aligned}
(111) = (222) &= \frac{1}{5}, (333) = 22, (144) = (244) = \frac{6}{5}, (155) = (266) = 0, \\
(166) = (255) &= \frac{8}{5}, (344) = \frac{44}{5}, (355) = (366) = \frac{88}{5}, (456) = \frac{64}{5}.
\end{aligned}$$

Proof. Case of F_4 -II. We have the following decomposition according to the structure of F_4 -II:

$$\mathfrak{g} = A_1^1 + A_1^2 + C_2 + \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3.$$

If we let

$$\mathfrak{b} = A_1^1 + A_1^2 + C_2 + \mathfrak{m}_1, \quad \mathfrak{p} = \mathfrak{m}_2 + \mathfrak{m}_3,$$

then $(\mathfrak{g}, \mathfrak{b})$ is an irreducible symmetric pair. In fact, this decomposition can be achieved by the first involutive automorphism θ , which means $\mathfrak{b} \cong B_4$, therefore simple. Now, we consider the following

metric on \mathfrak{g} :

$$\langle\langle \cdot, \cdot \rangle\rangle_1 = u_1 B|_{\mathfrak{b}} + u_2 B|_{\mathfrak{p}}, \quad (3.11)$$

this metric is the same as the one in (3.9) if we let $x_1 = x_2 = x_3 = x_4 = u_1$ and $x_5 = x_6 = u_2$. As a result, if we denote the components of the Ricci tensor with respect to (3.9) by \tilde{r}_1 and \tilde{r}_2 , then it holds $r_1 = r_2 = r_3 = r_4 = \tilde{r}_1$ and $r_5 = r_6 = \tilde{r}_2$. With comparing these equations, we have

$$\begin{aligned} \frac{1}{4d_1}((111) + (144)) &= \frac{1}{4d_2}((222) + (244)) = \frac{1}{4d_3}((333) + (344)) = \frac{1}{4} - \frac{1}{2d_4}(456), \\ \frac{1}{4d_1}((155) + (166)) &= \frac{1}{4d_2}((255) + (266)) = \frac{1}{4d_3}((355) + (366)) = \frac{1}{2d_4}(456). \end{aligned} \quad (3.12)$$

We consider the following decomposition of \mathfrak{g} ,

$$\mathfrak{g} = \mathfrak{b}' + \mathfrak{p}' = \mathfrak{b}'_1 + \mathfrak{b}'_2 + \mathfrak{p}',$$

where $\mathfrak{b}' = A_1^1 + A_1^1 + C_2 + \mathfrak{m}_2$, $\mathfrak{b}'_1 = A_1^1$, $\mathfrak{b}'_2 = A_1^2 + C_2 + \mathfrak{m}_2$, $\mathfrak{p} = \mathfrak{m}_1 + \mathfrak{m}_3$. In fact, $\mathfrak{b}'_2 \cong C_3$, and this decomposition is corresponding to the second involutive automorphism τ on \mathfrak{g} . Hence, $(\mathfrak{g}, \mathfrak{b}')$ is an irreducible symmetric pair. Now, we consider the metric on \mathfrak{g} as follows:

$$\langle\langle \cdot, \cdot \rangle\rangle_2 = w_1 B|_{A_1^1} + w_2 B|_{C_3} + w_3 B|_{\mathfrak{p}'}. \quad (3.13)$$

If we set $x_1 = w_1, x_2 = x_3 = x_5 = w_2, x_4 = x_6 = w_3$ in (3.9), then the two metrics are the same, as a result, we can get the following equations:

$$\begin{aligned} \frac{1}{4d_2}((222) + (255)) &= \frac{1}{4d_3}((333) + (355)) = \frac{1}{4} - \frac{1}{2d_5}((456) - (155)), \\ \frac{1}{4d_2}((244) + (266)) &= \frac{1}{4d_3}((344) + (366)) = \frac{1}{2d_5}(456), \\ \frac{1}{2d_5}(155) &= 0, \frac{1}{2d_4}(144) = \frac{1}{2d_6}(166). \end{aligned} \quad (3.14)$$

We can easily calculate $\alpha_{A_1}^{F_4} = \frac{8}{36}$ and $\alpha_{C_2}^{F_4} = \frac{12}{36}$ both with the long roots of F_4 , therefore, $(111) = (222) = \frac{2}{3}$ and $(333) = \frac{10}{3}$ according to Lemma 3.1, besides we get $(456) = \frac{20}{9}$ from [22]. Thus, along with the above equations (3.10), (3.12) and (3.14), one can easily get the solutions as the lemma, here $d_1 = 3, d_2 = 3, d_3 = 10, d_4 = 20, d_5 = 8, d_6 = 8$.

Case of E_8 -I. According to [12], E_8 -I has the following decomposition:

$$\mathfrak{g} = A_1^1 + A_1^2 + D_6 + \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3.$$

According to the two commutative involutive automorphisms on E_8 -I in [12], we have the following two decompositions which make $(\mathfrak{g}, \mathfrak{b})$ and $(\mathfrak{g}, \mathfrak{b}')$ be two different irreducible symmetric pairs.

$$\mathfrak{g} = \mathfrak{b} + \mathfrak{p}, \mathfrak{b} = A_1^1 + A_1^2 + D_6 + \mathfrak{m}_1 \cong D_8, \mathfrak{p} = \mathfrak{m}_2 + \mathfrak{m}_3,$$

$$\mathfrak{g} = \mathfrak{b}' + \mathfrak{p}', \mathfrak{b}' = \mathfrak{b}'_1 + \mathfrak{b}'_2 = A_1 + A_1 + D_6 + \mathfrak{m}_2, \mathfrak{b}'_1 = A_1^1, \mathfrak{b}'_2 = A_1^2 + D_6 + \mathfrak{m}_2 \cong E_7, \mathfrak{p}' = \mathfrak{m}_1 + \mathfrak{m}_3.$$

By using the same methods above, we can calculate $(111) = (222) = \frac{1}{5}$ and $(333) = 22$ by Lemma 3.1 and get $(456) = \frac{64}{5}$, as a result, we have

$$\begin{aligned} (111) &= (222) = \frac{1}{5}, (333) = 22, (144) = (244) = \frac{6}{5}, (155) = (266) = 0, \\ (166) &= (255) = \frac{8}{5}, (344) = \frac{44}{5}, (355) = (366) = \frac{88}{5}, (456) = \frac{64}{5}. \end{aligned}$$

□

For $p = 4$, there is only E_7 -I in this type, and the decomposition according to [12] is

$$\mathfrak{g} = A_1^1 + A_1^2 + A_1^3 + D_4 + \mathfrak{p}_1 + \mathfrak{p}_2 + \mathfrak{p}_3.$$

The considered metric is of the following form:

$$\langle , \rangle = x_1 B|_{A_1^1} + x_2 B|_{A_1^2} + x_3 B|_{A_1^3} + x_4 B|_{D_4} + x_5 B|_{\mathfrak{m}_1} + x_6 B|_{\mathfrak{m}_2} + x_7 B|_{\mathfrak{m}_3}, \quad (3.15)$$

It is easy to know that the possible non-zero coefficients in the expression for components of Ricci tensor with respect to the metric given by (3.15) are

$$(111), (222), (333), (444), (155), (166), (177), (255), (266), (277), (355), (366), (377), (455), (466), (477).$$

As a result, the components of Ricci tensor are:

$$\begin{aligned} r_1 &= \frac{1}{4d_1} \left(\frac{1}{x_1} (111) + \frac{x_1}{x_5^2} (155) + \frac{x_1}{x_6^2} (166) + \frac{x_1}{x_7^2} (177) \right), \\ r_2 &= \frac{1}{4d_2} \left(\frac{1}{x_2} (222) + \frac{x_2}{x_5^2} (255) + \frac{x_2}{x_6^2} (266) + \frac{x_2}{x_7^2} (277) \right), \\ r_3 &= \frac{1}{4d_3} \left(\frac{1}{x_3} (333) + \frac{x_3}{x_5^2} (355) + \frac{x_3}{x_6^2} (366) + \frac{x_3}{x_7^2} (377) \right), \\ r_4 &= \frac{1}{4d_4} \left(\frac{1}{x_4} (444) + \frac{x_4}{x_5^2} (455) + \frac{x_4}{x_6^2} (466) + \frac{x_4}{x_7^2} (477) \right), \\ r_5 &= \frac{1}{2x_5} + \frac{1}{2d_5} (567) \left(\frac{x_5}{x_6 x_7} - \frac{x_6}{x_7 x_5} - \frac{x_7}{x_5 x_6} \right) - \frac{1}{2d_5} \left(\frac{x_1}{x_5^2} (155) + \frac{x_2}{x_5^2} (255) + \frac{x_3}{x_5^2} (355) + \frac{x_4}{x_5^2} (455) \right), \\ r_6 &= \frac{1}{2x_6} + \frac{1}{2d_6} (567) \left(\frac{x_6}{x_5 x_7} - \frac{x_7}{x_6 x_5} - \frac{x_5}{x_7 x_6} \right) - \frac{1}{2d_6} \left(\frac{x_1}{x_6^2} (166) + \frac{x_2}{x_6^2} (266) + \frac{x_3}{x_6^2} (366) + \frac{x_4}{x_6^2} (466) \right), \\ r_7 &= \frac{1}{2x_7} + \frac{1}{2d_7} (567) \left(\frac{x_7}{x_5 x_6} - \frac{x_5}{x_6 x_7} - \frac{x_6}{x_7 x_6} \right) - \frac{1}{2d_7} \left(\frac{x_1}{x_7^2} (177) + \frac{x_2}{x_7^2} (277) + \frac{x_3}{x_7^2} (377) + \frac{x_4}{x_7^2} (477) \right). \end{aligned}$$

Lemma 3.4. *The possible non-zero coefficients in the expression for components of Ricci tensor with respect to the metric given by (3.15) are as follows:*

$$\begin{aligned} (111) &= (222) = (333) = \frac{1}{3}, (444) = \frac{28}{3}, (567) = \frac{64}{9}, \\ (166) &= (177) = (255) = (277) = (355) = (366) = \frac{4}{3}, \\ (155) &= (266) = (377) = 0, (455) = (466) = (477) = \frac{56}{9}. \end{aligned}$$

Proof. According to the two commutative involutive automorphisms on E_7 -I in [12], we have the following two decompositions which make $(\mathfrak{g}, \mathfrak{b})$ and $(\mathfrak{g}, \mathfrak{b}')$ be two different irreducible symmetric pairs.

$$\begin{aligned}\mathfrak{g} &= \mathfrak{b} + \mathfrak{p}, \mathfrak{b} = A_1^1 + A_1^2 + A_1^3 + D_4 + \mathfrak{m}_1 = \mathfrak{b}_1 + \mathfrak{b}_2, \\ \mathfrak{b}_1 &= A_1^1, \mathfrak{b}_2 = A_1^2 + A_1^3 + D_4 + \mathfrak{m}_1 \cong D_6, \mathfrak{p} = \mathfrak{m}_2 + \mathfrak{m}_3, \\ \mathfrak{g} &= \mathfrak{b}' + \mathfrak{p}', \mathfrak{b} = A_1^1 + A_1^2 + A_1^3 + D_4 + \mathfrak{m}_2 = \mathfrak{b}'_1 + \mathfrak{b}'_2, \\ \mathfrak{b}'_1 &= A_1^2, \mathfrak{b}'_2 = A_1^1 + A_1^3 + D_4 + \mathfrak{m}_2 \cong D_6, \mathfrak{p} = \mathfrak{m}_1 + \mathfrak{m}_3.\end{aligned}$$

Then we can use the similar methods to calculate out the coefficients given in the lemma. \square

Case of E_7 -II. In [12], E_7 -II has the following decomposition:

$$E_7 = T \oplus A_1 \oplus A_5 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3, \quad (3.16)$$

where T is the 1-dimensional center of \mathfrak{k} .

We consider left-invariant metrics on E_7 -II as follows:

$$\langle \cdot, \cdot \rangle = u_0 \cdot B|_T + x_1 \cdot B|_{A_1} + x_2 \cdot B|_{A_5} + x_3 \cdot B|_{\mathfrak{p}_1} + x_4 \cdot B|_{\mathfrak{p}_2} + x_5 \cdot B|_{\mathfrak{p}_3}, \quad (3.17)$$

where $u_0, x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}^+$. According to the structure of generalized Wallach space, the possible non-zero coefficients in the expression for the components of Ricci tensor with respect to the metric (3.17) are

$$(033), (044), (055), (111), (133), (144), (155), (222), (233), (244), (255), (345)$$

Therefore, by Lemma 2.2, the components of Ricci tensor with respect to the metric (3.17) can be expressed as follows:

$$\begin{cases} r_0 = \frac{1}{4d_0} \left(\frac{u_0}{x_3^2} (033) + \frac{u_0}{x_4^2} (044) + \frac{u_0}{x_5^2} (055) \right), \\ r_1 = \frac{1}{4d_1} \left(\frac{1}{x_1} (111) + \frac{x_1}{x_3^2} (133) + \frac{x_1}{x_4^2} (144) + \frac{x_1}{x_5^2} (155) \right), \\ r_2 = \frac{1}{4d_2} \left(\frac{1}{x_2} (222) + \frac{x_2}{x_3^2} (233) + \frac{x_2}{x_4^2} (244) + \frac{x_2}{x_5^2} (255) \right), \\ r_3 = \frac{1}{2x_3} + \frac{1}{2d_3} (345) \left(\frac{x_3}{x_4x_5} - \frac{x_4}{x_5x_3} - \frac{x_5}{x_3x_4} \right) - \frac{1}{2d_3} \left(\frac{u_0}{x_3^2} (033) + \frac{x_1}{x_3^2} (133) + \frac{x_2}{x_3^2} (233) \right), \\ r_4 = \frac{1}{2x_4} + \frac{1}{2d_4} (345) \left(\frac{x_4}{x_5x_3} - \frac{x_5}{x_3x_4} - \frac{x_3}{x_4x_5} \right) - \frac{1}{2d_4} \left(\frac{u_0}{x_4^2} (044) + \frac{x_1}{x_4^2} (144) + \frac{x_2}{x_4^2} (244) \right), \\ r_5 = \frac{1}{2x_5} + \frac{1}{2d_5} (345) \left(\frac{x_5}{x_3x_4} - \frac{x_3}{x_4x_5} - \frac{x_4}{x_5x_3} \right) - \frac{1}{2d_5} \left(\frac{u_0}{x_5^2} (055) + \frac{x_1}{x_5^2} (155) + \frac{x_2}{x_5^2} (255) \right). \end{cases}$$

and

$$\begin{aligned}
(033) + (044) + (055) &= d_0, \\
(111) + (133) + (144) + (155) &= d_1, \\
(222) + (233) + (244) + (255) &= d_2, \\
2(033) + 2(133) + 2(233) + 2(345) &= d_3, \\
2(044) + 2(144) + 2(244) + 2(345) &= d_4, \\
2(055) + 2(155) + 2(255) + 2(345) &= d_5,
\end{aligned} \tag{3.18}$$

where we used the symmetric properties of the indices in (ijk) .

Lemma 3.5. *In the case of E_7 -II, the possible non-zero coefficients in the expression for the components of Ricci tensor with respect to metric (3.17) are as follows:*

$$\begin{aligned}
(033) &= \frac{4}{9}, (044) = \frac{5}{9}, (055) = 0, (345) = \frac{20}{3}, \\
(111) &= \frac{1}{3}, (133) = 1, (144) = 0, (155) = \frac{5}{3}, \\
(222) &= \frac{35}{3}, (233) = \frac{35}{9}, (244) = \frac{70}{9}, (255) = \frac{35}{3}.
\end{aligned}$$

Proof. In fact, there are two involutive automorphisms on E_7 -II [12], denoted by σ and τ , where σ corresponds to the irreducible symmetric pair $(\mathfrak{g}, \mathfrak{b})$ having the following decomposition:

$$\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{p}, \quad \mathfrak{b} = \mathfrak{T} \oplus A_1 \oplus A_5 \oplus \mathfrak{p}_1 \cong A_7, \quad \mathfrak{p} = \mathfrak{p}_2 \oplus \mathfrak{p}_3, \tag{3.19}$$

and τ corresponds to the irreducible symmetric pair $(\mathfrak{g}, \mathfrak{b}')$ having the following decomposition:

$$\mathfrak{g} = \mathfrak{b}' \oplus \mathfrak{p}', \quad \mathfrak{b}' = \mathfrak{b}'_1 \oplus \mathfrak{b}'_2, \quad \mathfrak{b}'_1 = A_1, \quad \mathfrak{b}'_2 = \mathfrak{T} \oplus A_5 \oplus \mathfrak{p}_2 \cong D_6, \quad \mathfrak{p}' = \mathfrak{p}_1 \oplus \mathfrak{p}_3. \tag{3.20}$$

We consider the following left-invariant metric on E_7 with respect to the decomposition (3.19)

$$(\ , \)_1 = w_1 \cdot B|_{\mathfrak{b}} + w_2 \cdot B|_{\mathfrak{p}}, \tag{3.21}$$

if we let $u_0 = x_1 = x_2 = x_3 = w_1$ and $x_4 = x_5 = w_2$ in (3.17), then these two metrics are the same, as a result, if we denote the components of Ricci tensor with respect to the metric (3.21) by \tilde{r}_1 and \tilde{r}_2 , then we have $r_0 = r_1 = r_2 = r_3 = \tilde{r}_1$ and $r_4 = r_5 = \tilde{r}_2$. With an easy calculation we get the following equations of the possible non-zero coefficients:

$$\begin{aligned}
\frac{1}{4d_0}(033) &= \frac{1}{d_1}((111) + (133)) = \frac{1}{d_2}((222) + (233)) = \frac{1}{4} - \frac{1}{2d_3}(345), \\
\frac{1}{4d_0}((044) + (055)) &= \frac{1}{4d_1}((144) + (155)) = \frac{1}{4d_2}((244) + (255)) = \frac{1}{2d_3}(345),
\end{aligned} \tag{3.22}$$

where we used the equations in (3.18) for simplification.

On the other hand, we consider the left-invariant metrics on E_7 -II with respect to the decomposition (3.20) as follows:

$$(\ , \)_2 = v_1 \cdot B|_{A_1} + v_2 \cdot B|_{D_6} + v_3 \cdot B|_{\mathfrak{p}'}, \tag{3.23}$$

if we let $x_1 = v_1$, $u_0 = x_2 = x_4 = v_2$ and $x_3 = x_5 = v_3$ in the metric (3.17), then the components of Ricci tensor with respect to these two metrics are the same, which means if we denote the components of Ricci tensor with respect to the metric (3.23) by \tilde{r}'_1 , \tilde{r}'_2 and \tilde{r}'_3 , we have $r_1 = \tilde{r}'_1$, $r_0 = r_2 = r_4 = \tilde{r}'_2$ and $r_3 = r_5 = \tilde{r}'_3$. With a short calculation, we have the following equations:

$$\begin{aligned} \frac{1}{4d_0}(044) &= \frac{1}{4d_3}((222) + (244)) = \frac{1}{4} - \frac{1}{2d_4}(345), \\ \frac{1}{4d_0}((033) + (055)) &= \frac{1}{4d_2}((233) + (255)) = \frac{1}{2d_4}(345), \end{aligned} \quad (3.24)$$

where we used the equations in (3.18) for simplification.

By Lemma 3.1, we can calculate that $(111) = \frac{1}{3}$ and $(222) = \frac{35}{3}$, with Table 1 in [22], we have $(345) = \frac{20}{3}$ and $d_0 = 1, d_1 = 1, d_2 = 35, d_3 = 24, d_4 = 30, d_5 = 40$, along with linear equations (3.22) and (3.24), we get the possible non-zero coefficients as follows:

$$\begin{aligned} (033) &= \frac{4}{9}, (044) = \frac{5}{9}, (055) = 0, (345) = \frac{20}{3}, \\ (111) &= \frac{1}{3}, (133) = 1, (144) = 0, (155) = \frac{5}{3}, \\ (222) &= \frac{35}{3}, (233) = \frac{35}{9}, (244) = \frac{70}{9}, (255) = \frac{35}{3}. \end{aligned} \quad (3.25)$$

□

Case of E_6 -II. As is shown in [12], E_6 -II can be decomposed as follows:

$$\mathfrak{g} = \mathbb{T} \oplus A_1^1 \oplus A_1^2 \oplus A_3 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3, \quad (3.26)$$

and we consider the following left-invariant metrics on E_6 according to this decomposition:

$$\langle \cdot, \cdot \rangle = u_0 \cdot B|_{\mathbb{T}} + x_1 \cdot B|_{A_1^1} + x_2 \cdot B|_{A_1^2} + x_3 \cdot B|_{A_3} + x_4 \cdot B|_{\mathfrak{p}_1} + x_5 \cdot B|_{\mathfrak{p}_2} + x_6 \cdot B|_{\mathfrak{p}_3}, \quad (3.27)$$

where $u_0, x_1, x_2, x_3, x_4, x_5, x_6 \in \mathbb{R}^+$. Because of the structure of generalized Wallach space, the possible non-zero coefficients in the expressions for the components of Ricci tensor with respect to the metric (3.27) are as follows:

$$(044), (055), (066), (111), (144), (155), (166), (222), (244), (255), (266), (333), (344), (355), (366), (456).$$

By Lemma 2.2, the components of Ricci tensor with respect to the metric (3.27) can be simplified as follows:

$$\left\{ \begin{array}{l} r_0 = \frac{1}{4d_0} \left(\frac{u_0}{x_4^2}(044) + \frac{u_0}{x_5^2}(055) + \frac{u_0}{x_6^2}(066) \right), \\ r_1 = \frac{1}{4d_1} \left(\frac{1}{x_1}(111) + \frac{x_1}{x_4^2}(144) + \frac{x_1}{x_5^2}(155) + \frac{x_1}{x_6^2}(166) \right), \\ r_2 = \frac{1}{4d_2} \left(\frac{1}{x_2}(222) + \frac{x_2}{x_4^2}(244) + \frac{x_2}{x_5^2}(255) + \frac{x_2}{x_6^2}(266) \right), \\ r_3 = \frac{1}{4d_3} \left(\frac{1}{x_3}(333) + \frac{x_3}{x_4^2}(344) + \frac{x_3}{x_5^2}(355) + \frac{x_3}{x_6^2}(366) \right), \\ r_4 = \frac{1}{2x_4} + \frac{1}{2d_4}(456) \left(\frac{x_4}{x_5x_6} - \frac{x_5}{x_6x_4} - \frac{x_6}{x_4x_5} \right) - \frac{1}{2d_4} \left(\frac{u_0}{x_4^2}(044) + \frac{x_1}{x_4^2}(144) + \frac{x_2}{x_4^2}(244) + \frac{x_3}{x_4^2}(344) \right), \\ r_5 = \frac{1}{2x_5} + \frac{1}{2d_5}(456) \left(\frac{x_5}{x_6x_4} - \frac{x_6}{x_4x_5} - \frac{x_4}{x_5x_6} \right) - \frac{1}{2d_5} \left(\frac{u_0}{x_5^2}(055) + \frac{x_1}{x_5^2}(155) + \frac{x_2}{x_5^2}(255) + \frac{x_3}{x_5^2}(355) \right), \\ r_6 = \frac{1}{2x_6} + \frac{1}{2d_6}(456) \left(\frac{x_6}{x_4x_5} - \frac{x_4}{x_5x_6} - \frac{x_5}{x_6x_4} \right) - \frac{1}{2d_6} \left(\frac{u_0}{x_6^2}(066) + \frac{x_1}{x_6^2}(166) + \frac{x_2}{x_6^2}(266) + \frac{x_3}{x_6^2}(366) \right). \end{array} \right.$$

and

$$\begin{aligned} (044) + (055) + (066) &= d_0, \\ (111) + (144) + (155) + (166) &= d_1, \\ (222) + (244) + (255) + (266) &= d_2, \\ (333) + (344) + (355) + (366) &= d_3, \\ 2(044) + 2(144) + 2(244) + 2(344) + 2(456) &= d_4, \\ 2(055) + 2(155) + 2(255) + 2(266) + 2(456) &= d_5, \\ 2(066) + 2(166) + 2(266) + 2(366) + 2(456) &= d_6, \end{aligned} \tag{3.28}$$

where we used the symmetric properties of the indices in (ijk) .

Lemma 3.6. *In the case of E_6-II , the possible non-zero coefficients in the expression for the components of Ricci tensor with respect to metric (3.27) are as follows:*

$$\begin{aligned} (044) &= 1/2, (055) = 1/2, (066) = 0, (456) = 4, \\ (111) &= 1/2, (144) = 0, (155) = 1, (166) = 3/2, \\ (222) &= 1/2, (244) = 1, (255) = 0, (266) = 3/2, \\ (333) &= 5, (344) = 5/2, (355) = 5/2, (366) = 5. \end{aligned}$$

Proof. In fact, according to [12], there are two involutive automorphisms on E_6-II , denoted by σ and τ , each of which corresponds an irreducible symmetric pair. Further, σ corresponds to the irreducible symmetric pair $(\mathfrak{g}, \mathfrak{b})$ with the following decomposition:

$$\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{p}, \quad \mathfrak{b} = \mathfrak{b}_1 \oplus \mathfrak{b}_2, \quad \mathfrak{b}_1 = A_1^1, \quad \mathfrak{b}_2 = T \oplus A_1^2 \oplus A_3 \oplus \mathfrak{p}_1 \cong A_5, \quad \mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2, \tag{3.29}$$

while τ corresponds to the irreducible symmetric pair (\mathfrak{g}, \cdot) with decomposition as follows:

$$\mathfrak{g} = \mathfrak{b}' \oplus \mathfrak{p}', \quad \mathfrak{b}' = \mathfrak{b}'_1 \oplus \mathfrak{b}'_2, \quad \mathfrak{b}'_1 = A_1^2, \quad \mathfrak{b}'_2 = T \oplus A_1^1 \oplus A_3 \oplus \mathfrak{p}_2 \cong A_5, \quad \mathfrak{p}' = \mathfrak{p}_1 \oplus \mathfrak{p}_2. \quad (3.30)$$

Then, we consider the following left-invariant metrics on E_6 according to the decomposition (3.29):

$$(\cdot, \cdot)_1 = w_1 \cdot B|_{A_1^1} + w_2 \cdot B|_{A_5} + w_3 \cdot B|_{\mathfrak{p}}. \quad (3.31)$$

If we let $u_0 = x_2 = x_3 = x_4 = w_2$, $x_1 = w_1$ and $x_5 = x_6 = w_3$ in the metric (3.27), then these two metrics are the same, as a result, the components of Ricci tensor with respect to these two metrics are equal respectively, which means if we denote the components of Ricci tensor with respect to metric (3.31) by \tilde{r}_1, \tilde{r}_2 and \tilde{r}_3 , then $r_0 = r_2 = r_3 = r_4 = \tilde{r}_2, r_1 = \tilde{r}_1$ and $r_5 = r_6 = \tilde{r}_3$, with a short calculation, one can get the following system of equations:

$$\begin{aligned} \frac{1}{4d_0}(044) &= \frac{1}{4d_2}((222) + (244)) = \frac{1}{4d_3}((333) + (344)) = \frac{1}{4} - \frac{1}{2d_4}(456), \\ \frac{1}{4d_0}((055) + (066)) &= \frac{1}{4d_2}((255) + (266)) = \frac{1}{4d_3}((355) + (366)) = \frac{1}{2d_4}(456), \\ \frac{1}{d_5}(155) &= \frac{1}{d_6}(166), \quad (144) = 0, \end{aligned} \quad (3.32)$$

where we used the equations in (3.28) for simplification.

On the other hand, we consider the following left-invariant metric on E_6 with respect to the decomposition (3.30):

$$(\cdot, \cdot)_2 = v_1 \cdot B|_{A_1^2} + v_2 \cdot B|_{A_5} + v_3 \cdot B|_{\mathfrak{p}'}. \quad (3.33)$$

If we let $u_0 = x_1 = x_3 = x_5 = v_2$, $x_2 = v_1$ and $x_4 = x_6 = v_3$ in the metric (3.27), then these two metrics are the same, as a result, the components of Ricci tensor with respect to these two metrics are equal respectively, which means if we denote the components of Ricci tensor with respect to metric (3.31) by $\tilde{r}'_1, \tilde{r}'_2$ and \tilde{r}'_3 , then $r_0 = r_1 = r_3 = r_5 = \tilde{r}'_2, r_2 = \tilde{r}'_1$ and $r_4 = r_6 = \tilde{r}'_3$, with a short calculation, one can get the following system of equations:

$$\begin{aligned} \frac{1}{4d_0}(055) &= \frac{1}{4d_1}((111) + (155)) = \frac{1}{4d_3}((333) + (355)) = \frac{1}{4} - \frac{1}{2d_5}(456), \\ \frac{1}{4d_0}((044) + (066)) &= \frac{1}{4d_1}((144) + (166)) = \frac{1}{4d_3}((344) + (366)) = \frac{1}{2d_5}(456), \\ \frac{1}{d_6}(266) &= \frac{1}{d_4}(244), \quad (255) = 0, \end{aligned} \quad (3.34)$$

where we used the equations in (3.28) for simplification.

By Lemma 3.1, we have $(111) = (222) = \frac{1}{2}$ and $(333) = 5$, from Table 1 in [22], we get $(456) = 4$ and $d_0 = 1, d_1 = d_2 = 3, d_3 = 15, d_4 = d_5 = 16, d_6 = 24$, along with the equations (3.32) and (3.34), one can get the possible non-zero coefficients as given in the Lemma: \square

Remark 3.7. Besides E_6 -III, for the other cases there are isomorphisms between the subalgebras of \mathfrak{k} in the decomposition corresponding to the structure of generalized Wallach space, but these isomorphisms

don't affect the behavior of the Ricci tensor. In particular, one can verify that $\text{Ric}_{<, >}(\mathfrak{k}_i, \mathfrak{k}_j) = 0 (i \neq j)$, where \mathfrak{k}_i is isomorphism to \mathfrak{k}_j . As a result, Ric is still diagonal.

4. DISCUSSIONS ON NON-NATURALLY REDUCTIVE EINSTEIN METRICS

Now we have obtained the components of Ricci tensor for each case by Lemma 3.2, Lemma 3.3 and Lemma 3.4, from which we will find non-naturally reductive Einstein metrics case by case in this section.

Case of $p = 2$. We will give the criterion to determine whether a left-invariant metric of the form (3.1) is naturally reductive.

Proposition 4.1. If a left-invariant metric $<, >$ of the form (2.1) on G is naturally reductive with respect to $G \times L$ for some closed subgroup L of G , then for the case of $p = 2$, one of the following holds:

Case of $E_6\text{-III}$: 1) $x_2 = x_3, x_4 = x_5$ 2) $x_1 = x_2 = x_4, x_3 = x_5$ 3) $x_1 = x_2 = x_5, x_3 = x_4$ 4) $x_3 = x_4 = x_5$.

Case of $E_8\text{-II}$: 1) $x_1 = x_2 = x_3, x_4 = x_5$ 2) $x_1 = x_2 = x_4, x_3 = x_5$ 3) $x_1 = x_2 = x_5, x_3 = x_4$ 4) $x_3 = x_4 = x_5$.

Conversely, if one of 1), 2), 3), 4) is satisfied, then the metric of the form (3.1) is naturally reductive with respect to $G \times L$ for some closed subgroup L of G .

Proof. Let \mathfrak{l} be the Lie algebra of L . Then we have either $\mathfrak{l} \subset \mathfrak{k}$ or $\mathfrak{l} \not\subset \mathfrak{k}$. First we consider the case of $\mathfrak{l} \not\subset \mathfrak{k}$. Let \mathfrak{h} be the subalgebra of \mathfrak{g} generated by \mathfrak{l} and \mathfrak{k} . Since $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{k}_2 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ for the case of $p = 2$, \mathfrak{h} must contains only one of $\mathfrak{m}_1, \mathfrak{m}_2$ and \mathfrak{m}_3 . If $\mathfrak{m}_1 \subset \mathfrak{h}$, then $\mathfrak{k} \oplus \mathfrak{m}_1$ is a subalgebra of \mathfrak{g} according to [12]. In fact, for case of $E_6\text{-III}$, it is isomorphic to $A_1 \oplus A_5$, where $\mathfrak{k}_2 \oplus \mathfrak{m}_1 \cong A_5$. Therefore by Theorem 2.1, we have $x_2 = x_3, x_4 = x_5$. Similarly, for case of $E_8\text{-II}$, we have $x_1 = x_2 = x_3, x_4 = x_5$. By a similar way, we can obtain 2) and 3) in the proposition for each case.

Now we consider the case $\mathfrak{l} \subset \mathfrak{k}$. Since the orthogonal complement \mathfrak{l}^\perp of \mathfrak{l} with respect to B contains the orthogonal complement \mathfrak{k}^\perp of \mathfrak{k} , we see that $\mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \subset \mathfrak{l}^\perp$. Since the invariant metric $<, >$ is naturally reductive with respect to $G \times L$, it follows that $x_3 = x_4 = x_5$ by Theorem 2.1. The converse is a direct consequence of Theorem 2.1. \square

Case of $E_6\text{-III}$. According Lemma 3.2, we have

$$\begin{aligned} r_1 &= \frac{1}{12} \left(\frac{1}{2x_1} + \frac{7x_1}{4x_4^2} + \frac{3x_1}{4x_5^2} \right), \\ r_2 &= \frac{1}{84} \left(\frac{7}{x_2} + \frac{7x_2}{2x_3^2} + \frac{35x_2}{4x_4^2} + \frac{7x_2}{4x_5^2} \right), \\ r_3 &= \frac{1}{2x_3} + \frac{1}{8} \left(\frac{x_3}{x_4x_5} - \frac{x_4}{x_3x_5} - \frac{x_5}{x_3x_4} \right) - \frac{1}{8} \frac{x_2}{x_3^2}, \\ r_4 &= \frac{1}{2x_4} + \frac{1}{16} \left(\frac{x_4}{x_3x_5} - \frac{x_3}{x_4x_5} - \frac{x_5}{x_3x_4} \right) - \frac{1}{56} \left(\frac{7x_1}{4x_4^2} + \frac{35x_2}{4x_4^2} \right), \\ r_5 &= \frac{1}{2x_5} + \frac{7}{48} \left(\frac{x_5}{x_4x_3} - \frac{x_4}{x_3x_5} - \frac{x_3}{x_5x_4} \right) - \frac{1}{24} \left(\frac{3x_1}{4x_5^2} + \frac{7x_2}{4x_5^2} \right). \end{aligned}$$

We consider the system of equations

$$r_1 - r_2 = 0, r_2 - r_3 = 0, r_3 - r_4 = 0, r_4 - r_5 = 0. \quad (4.1)$$

Then finding Einstein metrics of the form (3.1) reduces to finding the positive solutions of system (4.1), and we normalize the equations by putting $x_5 = 1$. Then we obtain the system of equations:

$$\begin{aligned} g_1 &= 3x_1^2x_4^2x_2x_3^2 - x_1x_2^2x_3^2x_4^2 + 7x_1^2x_2x_3^2 - 5x_2^2x_1x_3^2 \\ &\quad - 2x_2^2x_1x_4^2 - 4x_1x_4^2x_3^2 + 2x_2x_4^2x_3^2 = 0, \\ g_2 &= x_2^2x_3^2x_4^2 - 6x_2x_3^3x_4 + 6x_2x_3x_4^3 + 5x_2^2x_3^2 \\ &\quad + 8x_2^2x_4^2 - 24x_2x_3x_4^2 + 4x_4^2x_3^2 + 6x_2x_3x_4 = 0, \\ g_3 &= 6x_3^3x_4 - 6x_4^3x_3 + x_1x_3^2 + 5x_2x_3^2 - 4x_2x_4^2 \\ &\quad - 16x_4x_3^2 + 16x_4^2x_3 - 2x_3x_4 = 0, \\ g_4 &= 3x_1x_4^2x_3 + 7x_2x_3x_4^2 + 8x_4x_3^2 - 48x_4^2x_3 \\ &\quad + 20x_4^3 - 3x_1x_3 - 15x_2x_3 + 48x_3x_4 - 20x_4 = 0. \end{aligned} \quad (4.2)$$

We consider a polynomial ring $R = \mathbb{Q}[z, x_1, x_2, x_3, x_4]$ and an ideal I generated by $\{g_1, g_2, g_3, g_4, zx_1x_2x_3x_4 - 1\}$ to find non-zero solutions of equations (4.2). We take a lexicographic order $>$ with $z > x_1 > x_2 > x_3 > x_4$ for a monomial ordering on R . Then with the aid of computer, the following polynomial is contained in the Gröbner basis for the ideal I

$$(x_4 - 1)(5x_4 - 3)(5x_4 - 19)h(x_4),$$

where $h(x_4)$ is of the form

$$\begin{aligned}
& 620527834748568712226625 x_4^{46} - 17142288459030942157682550 x_4^{45} \\
& + 253864478218386260238125175 x_4^{44} - 2655902456682476684982068196 x_4^{43} \\
& + 21712791139584353835485509485 x_4^{42} - 146403945857203174056695826906 x_4^{41} \\
& + 841869064160931856135565647035 x_4^{40} - 4221496823183250288515785683288 x_4^{39} \\
& + 18759663705905743422883726751607 x_4^{38} - 74772865457920384779255097278978 x_4^{37} \\
& + 269810159883362340386858437110705 x_4^{36} - 887921763230246899234386977810964 x_4^{35} \\
& + 2680934636604177050138806853307267 x_4^{34} - 7463374027396529763571324631419086 x_4^{33} \\
& + 19236417063475016928618367461353205 x_4^{32} - 46067840601646061767106985652544544 x_4^{31} \\
& + 102827687516709239196388118203922250 x_4^{30} - 214524790280392516306729959721167612 x_4^{29} \\
& + 419397032805870597215879980744285542 x_4^{28} - 770243915769947385884764661220911880 x_4^{27} \\
& + 1332121344976070900463276148279367906 x_4^{26} - 2174890268260219400375855326539637796 x_4^{25} \\
& + 3360427246597106223273731326789411118 x_4^{24} - 4926216667916456054561642964481111312 x_4^{23} \\
& + 6868691169547446497713714767840930254 x_4^{22} - 9130195555722867119052619469924183268 x_4^{21} \\
& + 11592691118354371158751244714914822658 x_4^{20} - 14080199932731634104314039587318940936 x_4^{19} \\
& + 16371489061442382163179799619487885062 x_4^{18} - 18224385478214023428533707109785905340 x_4^{17} \\
& + 19411619002013319176816159460886119530 x_4^{16} - 19763525846081914361620786672655930400 x_4^{15} \\
& + 19206776730802850981024169581516423525 x_4^{14} - 17785423401614562582040867258711520750 x_4^{13} \\
& + 15655271805998900455634624058209469875 x_4^{12} - 13053707551812208998459860496371252500 x_4^{11} \\
& + 10256813778567948010832363355641230625 x_4^{10} - 7536513577438512742744940874573881250 x_4^9 \\
& + 5123727743209309471365339473434734375 x_4^8 - 3177971936471765628211397123442375000 x_4^7 \\
& + 1766171205209537862901035966120796875 x_4^6 - 859452515630617218777390718317656250 x_4^5 \\
& + 355236019137782072058927844356328125 x_4^4 - 119504109722879595751903187339062500 x_4^3 \\
& + 30626968356732968210488474290234375 x_4^2 - 5304673599935287496386273558593750 x_4 \\
& + 463705449010204912215012369140625
\end{aligned} \tag{4.3}$$

In fact, in the Gröbner basis of the ideal I , x_1, x_2 and x_3 can be written into polynomials of x_4 . By solving $h(x_4) = 0$, we have four solutions, namely $0.6711159524, 0.8439629969, 0.9167404817, 2.171597540$ and all corresponding solutions of the system of equations $\{g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0, h(x_4) = 0\}$

with $x_1x_2x_3x_4 \neq 0$ are as follows:

$$\begin{aligned} &\{x_1 \approx 0.1550362575, x_2 \approx 0.7478555199, x_3 \approx 1.042517676, x_4 \approx 0.6711159524\}, \\ &\{x_1 \approx 1.366119112, x_2 \approx 0.2521439928, x_3 \approx 0.6761926469, x_4 \approx 0.8439629969\}, \\ &\{x_1 \approx 0.09898950458, x_2 \approx 0.2192177752, x_3 \approx 0.5309066187, x_4 \approx 0.9167404817\}, \\ &\{x_1 \approx 0.2432173551, x_2 \approx 0.4934849553, x_3 \approx 2.270522057, x_4 \approx 2.171597540\}. \end{aligned}$$

Due to Proposition 4.1, we conclude that each of these four solutions induces a non-naturally reductive Einstein metric.

For $x_4 = 1$, the system $\{g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0\}$ has the following four solutions:

$$\begin{aligned} &\{x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 1\}, \\ &\{x_1 \approx 0.1030504001, x_2 \approx 0.3244706112, x_3 \approx 0.3244706112, x_4 = 1\}, \\ &\{x_1 \approx 1.613068224, x_2 \approx 0.4009377358, x_3 \approx 0.4009377358, x_4 = 1\}, \\ &\{x_1 \approx 0.1984241041, x_2 \approx 1.108447830, x_3 \approx 1.108447830, x_4 = 1\}. \end{aligned}$$

For $x_4 = \frac{3}{5}$, the system $\{g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0\}$ has only one solution given by

$$\{x_1 = x_2 = x_4 = \frac{3}{5}, x_3 = 1\},$$

and for $x_4 = \frac{19}{5}$, the system $\{g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0\}$ also has only one solution given by

$$\{x_1 = x_2 = 1, x_3 = x_4 = \frac{19}{5}\}.$$

According to Proposition 4.1, these corresponding left-invariant Einstein metrics are all naturally reductive.

In summarize, we find 4 different non-naturally reductive left-invariant Einstein metrics on E_6 -III.

Case of E_8 -II. According Lemma 3.2, we have

$$\begin{aligned} r_1 &= \frac{1}{112} \left(\frac{28}{5x_1} + \frac{112x_1}{15x_3^2} + \frac{112x_1}{15x_4^2} + \frac{112x_1}{15x_5^2} \right), \\ r_2 &= \frac{1}{112} \left(\frac{28}{5x_2} + \frac{112x_2}{15x_3^2} + \frac{112x_2}{15x_4^2} + \frac{112x_2}{15x_5^2} \right), \\ r_3 &= \frac{1}{2x_3} + \frac{2}{15} \left(\frac{x_3}{x_4x_5} - \frac{x_4}{x_3x_5} - \frac{x_5}{x_3x_4} \right) - \frac{1}{128} \left(\frac{112x_1}{15x_3^2} + \frac{112x_2}{15x_3^2} \right), \\ r_4 &= \frac{1}{2x_4} + \frac{2}{15} \left(\frac{x_4}{x_3x_5} - \frac{x_3}{x_4x_5} - \frac{x_5}{x_3x_4} \right) - \frac{1}{128} \left(\frac{112x_1}{15x_4^2} + \frac{112x_2}{15x_4^2} \right), \\ r_5 &= \frac{1}{2x_5} + \frac{2}{15} \left(\frac{x_5}{x_4x_3} - \frac{x_4}{x_3x_5} - \frac{x_3}{x_5x_4} \right) - \frac{1}{128} \left(\frac{112x_1}{15x_5^2} + \frac{112x_2}{15x_5^2} \right), \end{aligned}$$

We consider the system of equations given by

$$r_1 - r_2 = 0, r_2 - r_3 = 0, r_3 - r_4 = 0, r_4 - r_5 = 0, \quad (4.4)$$

and normalize the equations by putting $x_5 = 1$, then we obtain the system of equations:

$$\begin{aligned}
g_1 &= 4x_1^2x_3^2x_4^2x_2 - 4x_1x_2^2x_3^2x_4^2 + 4x_1^2x_3^2x_2 + 4x_1^2x_4^2x_2 \\
&\quad - 4x_2^2x_1x_3^2 - 4x_2^2x_1x_4^2 - 3x_1x_3^2x_4^2 + 3x_2x_3^2x_4^2 = 0, \\
g_2 &= 8x_2^2x_3^2x_4^2 - 16x_3^3x_2x_4 + 16x_4^3x_2x_3 + 7x_1x_2x_4^2 + 8x_2^2x_3^2 \\
&\quad + 15x_2^2x_4^2 - 60x_2x_3x_4^2 + 6x_3^2x_4^2 + 16x_2x_3x_4 = 0, \\
g_3 &= 32x_3^3x_4 - 32x_4^3x_3 + 7x_1x_3^2 - 7x_1x_4^2 + 7x_2x_3^2 - 7x_2x_4^2 \\
&\quad - 60x_4x_3^2 + 60x_3x_4^2 = 0, \\
g_4 &= 7x_1x_3x_4^2 + 7x_2x_3x_4^2 - 60x_3x_4^2 + 32x_4^3 - 7x_1x_3 \\
&\quad - 7x_2x_3 + 60x_3x_4 - 32x_4 = 0.
\end{aligned} \tag{4.5}$$

We consider a polynomial ring $R = \mathbb{Q}[z, x_1, x_2, x_3, x_4]$ and an ideal I generated by $\{g_1, g_2, g_3, g_4, zx_1x_2x_3x_4 - 1\}$ to find non-zero solutions of equations (4.5). We take a lexicographic order $>$ with $z > x_1 > x_2 > x_3 > x_4$ for a monomial ordering on R . Then with the aid of computer, the following polynomial is contained in the Gröbner basis for the ideal I

$$(x_4 - 1)(23x_4 - 7)(7x_4 - 23)h(x_4),$$

where

$$\begin{aligned}
h(x_4) &= 18820892214681403392x_4^{24} - 106573710368905887744x_4^{23} \\
&\quad + 367021480848929587200x_4^{22} - 989697149383674494976x_4^{21} \\
&\quad + 1859094664559751753728x_4^{20} - 3257511072225679640576x_4^{19} \\
&\quad + 4280088309639423272992x_4^{18} - 5679995572440505667140x_4^{17} \\
&\quad + 6595970829340416842428x_4^{16} - 7511322681489787363579x_4^{15} \\
&\quad + 9419511263909486275350x_4^{14} - 9260548321425771133485x_4^{13} \\
&\quad + 11189718816841142104820x_4^{12} - 9260548321425771133485x_4^{11} \\
&\quad + 9419511263909486275350x_4^{10} - 7511322681489787363579x_4^9 \\
&\quad + 6595970829340416842428x_4^8 - 5679995572440505667140x_4^7 \\
&\quad + 4280088309639423272992x_4^6 - 3257511072225679640576x_4^5 \\
&\quad + 1859094664559751753728x_4^4 - 989697149383674494976x_4^3 \\
&\quad + 367021480848929587200x_4^2 - 106573710368905887744x_4 \\
&\quad + 18820892214681403392.
\end{aligned}$$

By solving $h(x_4) = 0$ numerically, we find positive four solutions which are given approximately by $x_4 \approx 0.3526915707$ (we state this solution will make x_2 negative), $x_4 \approx 0.7261283537$, $x_4 \approx 2.835338531$

and $x_4 \approx 1.377166991$ and we split the corresponding solutions of the system of equations $\{g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0, h(x_4) = 0\}$ with $x_1 x_2 x_3 x_4 \neq 0$ into two groups as follows:

$$\text{Group 1. } \begin{cases} \{x_1 \approx 1.304885525, x_2 \approx 0.4602586724, x_3 \approx 2.835338531, x_4 \approx 2.835338531\}, \\ \{x_1 \approx 0.4602586724, x_2 \approx 1.304885525, x_3 \approx 2.835338531, x_4 \approx 2.835338531\}. \end{cases}$$

$$\text{Group 2. } \begin{cases} \{x_1 \approx 0.1431443064, x_2 \approx 0.1431443064, x_3 = 1, x_4 \approx 0.7261283537\}, \\ \{x_1 \approx 0.1971336881, x_2 \approx 0.1971336881, x_3 \approx 1.377166991, x_4 \approx 1.377166991\}. \end{cases}$$

For $x_4 = 1$, the system $\{g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0\}$ has five solutions which can be split into the following three groups:

$$\text{Group 3. } \{x_1 = x_2 = x_3 = x_4 = 1\},$$

$$\text{Group 4. } \{x_1 = x_2 = x_3 = \frac{7}{23}, x_4 = 1\},$$

$$\text{Group 5. } \begin{cases} \{x_1 \approx 0.4602221254, x_2 \approx 0.1623293541, x_3 \approx 0.3526915707, x_4 = 1\}, \\ \{x_1 \approx 0.1623293541, x_2 \approx 0.4602221254, x_3 \approx 0.3526915707, x_4 = 1\}. \end{cases}$$

$$\text{Group 6. } \{x_1 \approx 0.1431443064, x_2 \approx 0.1431443064, x_3 \approx 0.7261283537, x_4 = 1\}.$$

For $x_4 = \frac{7}{23}$ and $x_4 = \frac{23}{7}$, the corresponding solutions of the system of equations $\{g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0\}$ are as follows respectively:

$$\text{Group 7. } \begin{cases} \{x_1 = x_2 = x_4 = \frac{7}{23}, x_3 = 1\}, \\ \{x_1 = x_2 = 1, x_3 = x_4 = \frac{23}{7}\}. \end{cases}$$

Among these solutions, we remark that the solution in Group 3 induces the Killing metric, the solutions in Group 4 and Group 7 induce the same metrics up to isometry which are naturally reductive due to Proposition 4.1, while the solutions in Group 1 and Group 5 induce the same metrics and the solutions in Group 2 and Group 6 also induce the same metrics up to isometry which are all non-naturally reductive due to Proposition 4.1.

Therefore, we find 2 non-naturally reductive Einstein metrics on $E_8\text{-II}$.

Case of $p = 3$. With the similar reason, we give the following proposition to decide whether a left-invariant metric is naturally reductive.

Proposition 4.2. If a left-invariant metric \langle, \rangle of the form (2.1) on G is naturally reductive with respect to $G \times L$ for some closed subgroup L of G , then for the case of $p = 3$, one of the following holds: 1) $x_1 = x_2 = x_3 = x_4, x_5 = x_6$ 2) $x_2 = x_3 = x_5, x_4 = x_6$ 3) $x_1 = x_3 = x_6, x_4 = x_5$ 4) $x_4 = x_5 = x_6$.

Conversely, if one of 1), 2), 3), 4) is satisfied, then the metric of the form (3.9) for the case of $p = 3$ is naturally reductive with respect to $G \times L$ for some closed subgroup L of G .

Case of F_4 -II. According Lemma 3.3, we have

$$\begin{aligned}
r_1 &= \frac{1}{12} \left(\frac{2}{3x_1} + \frac{5x_1}{3x_4^2} + \frac{2x_1}{3x_6^2} \right), \\
r_2 &= \frac{1}{12} \left(\frac{2}{3x_2} + \frac{5x_2}{3x_4^2} + \frac{2x_2}{3x_5^2} \right), \\
r_3 &= \frac{1}{40} \left(\frac{10}{3x_3} + \frac{40x_3}{9x_4^2} + \frac{10x_3}{9x_5^2} + \frac{10x_3}{9x_6^2} \right), \\
r_4 &= \frac{1}{2x_4} + \frac{1}{18} \left(\frac{x_4}{x_5x_6} - \frac{x_5}{x_6x_4} - \frac{x_6}{x_4x_5} \right) - \frac{1}{40} \left(\frac{5x_1}{3x_4^2} + \frac{5x_2}{3x_4^2} + \frac{40x_3}{9x_4^2} \right), \\
r_5 &= \frac{1}{2x_5} + \frac{5}{36} \left(\frac{x_5}{x_6x_4} - \frac{x_6}{x_4x_5} - \frac{x_4}{x_5x_6} \right) - \frac{1}{16} \left(\frac{2x_2}{3x_5^2} + \frac{10x_3}{9x_5^2} \right), \\
r_6 &= \frac{1}{2x_6} + \frac{5}{36} \left(\frac{x_6}{x_5x_4} - \frac{x_5}{x_6x_4} - \frac{x_4}{x_5x_6} \right) - \frac{1}{16} \left(\frac{2x_1}{3x_6^2} + \frac{9x_3}{9x_6^2} \right).
\end{aligned}$$

We consider the system of equations

$$r_1 - r_2 = 0, r_2 - r_3 = 0, r_3 - r_4 = 0, r_4 - r_5 = 0, r_5 - r_6 = 0. \quad (4.6)$$

Then finding Einstein metrics of the form (3.9) reduces to finding the positive solutions of system (4.6),

and we normalize the equations by putting $x_4 = 1$. Then we obtain the system of equations:

$$\begin{aligned}
g_1 &= 5x_1^2x_2x_5^2x_6^2 - 5x_1x_2^2x_5^2x_6^2 + 2x_1^2x_2x_5^2 - 2x_1x_2^2x_6^2 - 2x_1x_5^2x_6^2 + 2x_2x_5^2x_6^2 = 0, \\
g_2 &= 5x_2^2x_3x_5^2x_6^2 - 4x_2x_3^2x_5^2x_6^2 + 2x_2^2x_3x_6^2 - x_2x_3^2x_5^2 - x_2x_3^2x_6^2 - 3x_2x_5^2x_6^2 \\
&\quad + 2x_3x_5^2x_6^2 = 0, \\
g_3 &= -3x_1x_5^2x_6 - 3x_2x_5^2x_6 - 8x_3x_5^2x_6 - 14x_5^3 + 36x_5^2x_6 + 6x_5x_6^2 + 3x_2x_6 + 5x_3x_6 \\
&\quad - 36x_5x_6 + 14x_5 = 0, \\
g_4 &= -14x_4^3x_5 + 14x_4x_5^3 + 3x_1x_5^2 - 3x_2x_4^2 + 3x_2x_5^2 - 5x_3x_4^2 + 8x_3x_5^2 \\
&\quad + 36x_4^2x_5 - 36x_4x_5^2 - 6x_4x_5 = 0, \\
g_5 &= 20x_5^3x_6 - 20x_5x_6^3 + 3x_1x_5^2 - 3x_2x_6^2 + 5x_3x_5^2 - 5x_3x_6^2 - 36x_5^2x_6 + 36x_5x_6^2 = 0.
\end{aligned} \quad (4.7)$$

We consider a polynomial ring $R = Q[z, x_1, x_2, x_3, x_5, x_6]$ and an ideal I generated by $\{g_1, g_2, g_3, g_4, g_5, zx_1x_2x_3x_5x_6 - 1\}$ to find non-zero solutions of equations (4.7). We take a lexicographic order $>$ with $z > x_1 > x_2 > x_3 > x_5 > x_6$ for a monomial ordering on R . Then with the help of computer, the following polynomial is contained in the Gröbner basis for the ideal I

$$(x_6 - 1)(7x_6 - 11)(2375x_6^3 - 4195x_6^2 + 1960x_6 - 272) \cdot h(x_6), \quad (4.8)$$

where $h(x_6)$ is a polynomial of x_6 of degree 114. We put it in Appendix I for readers' convenience.

For $x_6 = 1$, with the polynomials in Gröbner basis, we get four values of x_5 , namely 0.2797176824, 0.3650688296, 1, 1.121529277, whose corresponding solutions of the system $\{g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0, g_5 = 0\}$ can be given as follows:

$$\text{Group 1. } \{x_1 = x_2 = x_3 = x_5 = x_6 = 1\}.$$

$$\text{Group 2.} \begin{cases} \{x_1 \approx 0.1355974584, x_2 = x_3 = x_5 \approx 0.2797176824, x_6 = 1\}, \\ \{x_1 \approx 1.653201132, x_2 = x_3 = x_5 \approx 0.3650688296, x_6 = 1\}, \\ \{x_1 \approx 0.2762168891, x_2 = x_3 = x_5 \approx 1.121529277, x_6 = 1\}. \end{cases}$$

For $x_6 = \frac{11}{7}$, we substitute it into the polynomials in the Gröbner basis, then we get the following corresponding solutions of the system $\{g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0, g_5 = 0\}$ with $x_1 x_2 x_3 x_5 x_6 \neq 0$:

$$\text{Group 3.} \{x_1 = x_2 = x_3 = 1, x_5 = x_6 = \frac{11}{7}\}.$$

By solving $x_6 - 1)(7x_6 - 11)(2375x_6^3 - 4195x_6^2 + 1960x_6 - 272 = 0$ numerically, there are three positive solutions given approximately by $x_6 \approx 0.2797176824$, $x_6 \approx 0.3650688296$ and $x_6 \approx 1.121529277$, whose corresponding solutions of the system $\{g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0, g_5 = 0, x_6 - 1)(7x_6 - 11)(2375x_6^3 - 4195x_6^2 + 1960x_6 - 272 = 0\}$ with $x_1 x_2 x_3 x_5 x_6 \neq 0$ are:

$$\text{Group 3.} \begin{cases} \{x_2 \approx 0.1355974584, x_1 = x_3 = x_6 \approx 0.2797176824, x_5 = 1\}, \\ \{x_2 \approx 1.653201132, x_1 = x_3 = x_6 \approx 0.3650688296, x_5 = 1\}, \\ \{x_2 \approx 0.2762168891, x_1 = x_3 = x_6 \approx 1.121529277, x_5 = 1\}. \end{cases}$$

By solving $h(x_6) = 0$ numerically, we get 6 different positive solutions, namely 0.4941864913, 0.7403305751, 1.068217773, 1.160571982, 1.345214992, 1.422410517, whose corresponding solutions of the system $\{g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0, g_5 = 0, h(x_6) = 0\}$ with $x_1 x_2 x_3 x_5 x_6 \neq 0$ can be split into the following three groups:

$$\text{Group 4.} \begin{cases} \{x_1 \approx 0.1516435461, x_2 \approx 0.1404443065, x_3 \approx 0.2282956381, x_5 \approx 1.068217773, x_6 \approx 0.4941864913\}, \\ \{x_1 \approx 0.1404443065, x_2 \approx 0.1516435461, x_3 \approx 0.2282956381, x_5 \approx 0.4941864913, x_6 \approx 1.068217773\}. \end{cases}$$

$$\text{Group 5.} \begin{cases} \{x_1 \approx 0.1951256737, x_2 \approx 1.654400436, x_3 \approx 0.3012253093, x_5 \approx 1.160571982, x_6 \approx 0.7403305751\}, \\ \{x_1 \approx 1.654400436, x_2 \approx 0.1951256737, x_3 \approx 0.3012253093, x_5 \approx 0.7403305751, x_6 \approx 1.160571982\}. \end{cases}$$

$$\text{Group 6.} \begin{cases} \{x_1 \approx 0.3075015814, x_2 \approx 1.094420015, x_3 \approx 1.138681932, x_5 \approx 1.422410516, x_6 \approx 1.345214992\}, \\ \{x_1 \approx 1.094420015, x_2 \approx 0.3075015814, x_3 \approx 1.138681932, x_5 \approx 1.345214992, x_6 \approx 1.422410516\}. \end{cases}$$

Among these metrics, we remark that the left-invariant Einstein metrics induced by the solutions in Group 1, 2, 3 are all naturally reductive, while the left-invariant Einstein metrics induced by the solutions in Group 4, 5, 6 are all non-naturally reductive due to Proposition 4.2. In particular, the solutions in Group 2 and 3 induce the same metrics up to isometry respectively and the solutions in each of Group 4-6 induce a same metric up to isometry.

In conclusion, we find 3 different non-naturally reductive left-invariant Einstein metrics on $F_4\text{-II}$.

Case of E_8 -I. According Lemma 3.3, we have

$$\begin{aligned}
r_1 &= \frac{1}{12} \left(\frac{1}{5x_1} + \frac{6x_1}{5x_4^2} + \frac{8x_1}{5x_6^2} \right), \\
r_2 &= \frac{1}{12} \left(\frac{1}{5x_2} + \frac{6x_2}{5x_4^2} + \frac{8x_2}{5x_5^2} \right), \\
r_3 &= \frac{1}{264} \left(\frac{22}{x_3} + \frac{44x_3}{5x_4^2} + \frac{88x_3}{5x_5^2} + \frac{88x_3}{5x_6^2} \right), \\
r_4 &= \frac{1}{2x_4} + \frac{2}{15} \left(\frac{x_4}{x_5x_6} - \frac{x_5}{x_6x_4} - \frac{x_6}{x_4x_5} \right) - \frac{1}{96} \left(\frac{6x_1}{5x_4^2} + \frac{6x_2}{5x_4^2} + \frac{44x_3}{5x_4^2} \right), \\
r_5 &= \frac{1}{2x_5} + \frac{1}{10} \left(\frac{x_5}{x_6x_4} - \frac{x_6}{x_4x_5} - \frac{x_4}{x_5x_6} \right) - \frac{1}{128} \left(\frac{8x_2}{5x_5^2} + \frac{88x_3}{88x_5^2} \right), \\
r_6 &= \frac{1}{2x_6} + \frac{1}{10} \left(\frac{x_6}{x_5x_4} - \frac{x_5}{x_6x_4} - \frac{x_4}{x_5x_6} \right) - \frac{1}{128} \left(\frac{8x_1}{5x_6^2} + \frac{88x_3}{5x_6^2} \right).
\end{aligned} \tag{4.9}$$

We consider the system of equations

$$r_1 - r_2 = 0, r_2 - r_3 = 0, r_3 - r_4 = 0, r_4 - r_5 = 0, r_5 - r_6 = 0. \tag{4.10}$$

Then finding Einstein metrics of the form (3.9) reduces to finding the positive solutions of system (4.10),

and we normalize the equations by putting $x_4 = 1$. Then we obtain the system of equations:

$$\begin{aligned}
g_1 &= 6x_1^2x_2x_5^2x_6^2 - 6x_1x_2^2x_5^2x_6^2 + 8x_1^2x_2x_5^2 - 8x_1x_2^2x_6^2 - x_1x_5^2x_6^2 + x_2x_5^2x_6^2 = 0, \\
g_2 &= 6x_2^2x_3x_5^2x_6^2 - 2x_2x_3^2x_5^2x_6^2 + 8x_2^2x_3x_6^2 - 4x_2x_3^2x_5^2 - 4x_2x_3^2x_6^2 - 5x_2x_5^2x_6^2 \\
&\quad + x_3x_5^2x_6^2 = 0, \\
g_3 &= 3x_1x_3x_5^2x_6^2 + 3x_2x_3x_5^2x_6^2 + 30x_3^2x_5^2x_6^2 + 32x_3x_5^3x_6 - 120x_3x_5^2x_6^2 + 32x_3x_5x_6^3 \\
&\quad + 16x_3^2x_5^2 + 16x_3^2x_6^2 + 20x_5^2x_6^2 - 32x_3x_5x_6 = 0, \\
g_4 &= -3x_1x_5^2x_6 - 3x_2x_5^2x_6 - 22x_3x_5^2x_6 - 56x_5^3 + 120x_5^2x_6 - 8x_5x_6^2 + 3x_2x_6 \\
&\quad + 33x_3x_6 - 120x_5x_6 + 56x_5 = 0, \\
g_5 &= 16x_5^3x_6 - 16x_5x_6^3 + x_1x_5^2 - x_2x_6^2 + 11x_3x_5^2 - 11x_3x_6^2 - 40x_5^2x_6 + 40x_5x_6^2 = 0.
\end{aligned} \tag{4.11}$$

We consider a polynomial ring $R = \mathbb{Q}[z, x_1, x_2, x_3, x_5, x_6]$ and an ideal I generated by $\{g_1, g_2, g_3, g_4, g_5, zx_1x_2x_3x_5x_6 - 1\}$ to find non-zero solutions of equations (4.11). We take a lexicographic order $>$ with $z > x_1 > x_2 > x_3 > x_5 > x_6$ for a monomial ordering on R . Then with the aid of computer, the following polynomial is contained in the Gröbner basis for the ideal I

$$(x_6 - 1)(7x_6 - 23)(864x_6^3 - 1676x_6^2 + 973x_6 - 177) \cdot f(x_6) \cdot h(x_6),$$

where $f(x_6)$ is a polynomial of x_6 given by

$$\begin{aligned}
f(x_6) &= 24313968x_6^{14} - 271810080x_6^{13} + 1334881896x_6^{12} - 4102312320x_6^{11} + 9388266607x_6^{10} \\
&\quad - 17066486910x_6^9 + 25201149031x_6^8 - 30982882320x_6^7 + 31894938304x_6^6 - 27360921600x_6^5 \\
&\quad + 19523164352x_6^4 - 11276897280x_6^3 + 5059512320x_6^2 - 1663672320x_6 + 301113344,
\end{aligned}$$

and $h(x_6)$ is a polynomial of x_6 of degree 114. For readers' convenience, we put it in Appendix II.

For $x_6 = 1$, from the Gröbner basis of the ideal I and with the aid of computer, we get the four solutions of the system $\{g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0, g_5 = 0\}$ which can be split into the following groups:

$$\begin{aligned} & \text{Group 1. } \{x_1 = x_2 = x_3 = x_4 = x_5 = 1\}, \\ & \text{Group 2. } \begin{cases} \{x_2 = x_3 = x_5 \approx 0.4188876552, x_1 \approx 0.04273408738, x_6 = 1\}, \\ \{x_2 = x_3 = x_5 \approx 0.4617244620, x_1 \approx 1.543913333, x_6 = 1\}, \\ \{x_2 = x_3 = x_5 \approx 1.059202697, x_1 \approx 0.07225088447, x_6 = 1\}. \end{cases} \end{aligned}$$

By solving $(7x_6 - 23)(864x_6^3 - 1676x_6^2 + 973x_6 - 177 = 0)$ numerically, there are three positive solutions which can be given approximately $x_6 \approx 0.4188876553, x_6 \approx 0.4617244621, x_6 \approx 1.059202697$ and the corresponding solutions of the system $\{g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0, g_5 = 0, 7x_6 - 23)(864x_6^3 - 1676x_6^2 + 973x_6 - 177 = 0\}$ with $x_1x_2x_3x_5x_6 \neq 0$ are given by

$$\text{Group 3. } \begin{cases} \{x_1 = x_3 = x_6 \approx 0.4188876552, x_2 \approx 0.04273408738, x_5 = 1\}, \\ \{x_1 = x_3 = x_6 \approx 0.4617244620, x_2 \approx 1.543913333, x_5 = 1\}, \\ \{x_1 = x_3 = x_6 \approx 1.059202697, x_2 \approx 0.07225088447, x_5 = 1\}. \end{cases}$$

By solving $f(x_6) = 0$ numerically, there are 6 different positive solutions, namely 0.7920673406, 0.8040419514, 1.075965351, 1.681651936, 2.596366999, 3.419732659, and the corresponding solutions of the system $\{g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0, g_5 = 0, f(x_6) = 0\}$ with $x_1x_2x_3x_5x_6 \neq 0$ are given by

$$\text{Group 3. } \begin{cases} \{x_1 = x_2 \approx 1.211722573, x_3 \approx 0.2521819866, x_5 = x_6 \approx 0.7920673405\}, \\ \{x_1 = x_2 \approx 0.04116638566, x_3 \approx 0.2299652722, x_5 = x_6 \approx 0.8040419514\}, \\ \{x_1 = x_2 \approx 0.07360068971, x_3 \approx 1.138692978, x_5 = x_6 \approx 1.075965351\}, \\ \{x_1 = x_2 \approx 0.07189340238, x_3 \approx 0.3957889206, x_5 = x_6 \approx 1.681651936\}, \\ \{x_1 = x_2 \approx 1.241147181, x_3 \approx 0.6544562607, x_5 = x_6 \approx 2.596366998\}, \\ \{x_1 = x_2 \approx 0.1594378743, x_3 \approx 1.292216476, x_5 = x_6 \approx 3.419732659\}. \end{cases}$$

By solving $h(x_6) = 0$ numerically, there are 10 different positive solutions, namely 0.4271280200, 0.4742936355, 0.7058209689, 0.8630215200, 1.008898001, 1.010769751, 2.058282527, 2.099884282, 3.402931725, 3.413270469, and the corresponding solutions of the system $\{g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0, g_5 = 0, f(x_6) = 0\}$ with $x_1x_2x_3x_5x_6 \neq 0$ can be split into the following 5 groups:

$$\begin{aligned} & \text{Group 4. } \begin{cases} \{x_1 \approx 0.0468426418, x_2 \approx 0.0433223582, x_3 \approx 0.4600814315, x_5 \approx 1.008898001, x_6 \approx 0.4271280200\}, \\ \{x_1 \approx 0.0433223582, x_2 \approx 0.0468426418, x_3 \approx 0.4600814315, x_5 \approx 0.4271280200, x_6 \approx 1.008898001\}. \end{cases} \\ & \text{Group 5. } \begin{cases} \{x_1 \approx 0.05053229100, x_2 \approx 1.535627653, x_3 \approx 0.5100903370, x_5 \approx 1.01076975, x_6 \approx 0.4742936355\}, \\ \{x_1 \approx 1.535627653, x_2 \approx 0.05053229100, x_3 \approx 0.5100903370, x_5 \approx 0.4742936355, x_6 \approx 1.01076975\}. \end{cases} \\ & \text{Group 6. } \begin{cases} \{x_1 \approx 1.075832320, x_2 \approx 0.04173283859, x_3 \approx 0.2381776636, x_5 \approx 0.8630215199, x_6 \approx 0.7058209688\}, \\ \{x_1 \approx 0.04173283859, x_2 \approx 1.075832320, x_3 \approx 0.2381776636, x_5 \approx 0.7058209688, x_6 \approx 0.8630215199\}. \end{cases} \\ & \text{Group 7. } \begin{cases} \{x_1 \approx 0.0887989484, x_2 \approx 1.442031123, x_3 \approx 0.4977693932, x_5 \approx 2.099884281, x_6 \approx 2.058282526\}, \\ \{x_1 \approx 1.442031123, x_2 \approx 0.0887989484, x_3 \approx 0.4977693932, x_5 \approx 2.058282526, x_6 \approx 2.099884281\}. \end{cases} \end{aligned}$$

$$\text{Group 8. } \begin{cases} \{x_1 \approx 0.1570295299, x_2 \approx 0.9524941307, x_3 \approx 1.170481952, x_5 \approx 3.413270468, x_6 \approx 3.402931724\}, \\ \{x_1 \approx 0.9524941307, x_2 \approx 0.1570295299, x_3 \approx 1.170481952, x_5 \approx 3.402931724, x_6 \approx 3.413270468\}. \end{cases}$$

Among these solutions, we remark that the solution in Group 1 is Killing metric, the solutions in Group 2 and Group 3 induce the same metrics up to isometry respectively which are naturally reductive due to Proposition 4.2, while the solutions in Group 3 induce 6 different left-invariant Einstein metrics which are non-naturally reductive and the solutions in each of Group 4-8 induce a same metric up to isometry which is non-naturally reductive due to Proposition 4.2.

In conclusion, we find 11 different non-naturally reductive Einstein metrics on E_8 -I.

Case of $p = 4$. By the same discussion in 4.1, we give the criterion to determine whether a left-invariant metric of the form (3.15) is naturally reductive.

Proposition 4.3. If a left-invariant metric \langle, \rangle of the form (2.1) on G is naturally reductive with respect to $G \times L$ for some closed subgroup L of G , then for the case of $p = 4$, one of the following holds: 1) $x_2 = x_3 = x_4 = x_5, x_6 = x_7$ 2) $x_1 = x_3 = x_4 = x_6, x_5 = x_7$ 3) $x_1 = x_2 = x_4 = x_7, x_5 = x_6$ 4) $x_5 = x_6 = x_7$.

Conversely, if one of 1), 2), 3), 4) is satisfied, then the metric of the form (3.15) for the case of $p = 4$ is naturally reductive with respect to $G \times L$ for some closed subgroup L of G .

Case of E_7 -I. Due to Lemma 3.4, we have the following equations:

$$\begin{aligned} r_1 &= \frac{1}{12} \left(\frac{1}{3x_1} + \frac{4x_1}{3x_6^2} + \frac{4x_1}{3x_7^2} \right), \\ r_2 &= \frac{1}{12} \left(\frac{1}{3x_2} + \frac{4x_2}{3x_5^2} + \frac{4x_2}{3x_7^2} \right), \\ r_3 &= \frac{1}{12} \left(\frac{1}{3x_3} + \frac{4x_3}{3x_5^2} + \frac{4x_3}{3x_6^2} \right), \\ r_4 &= \frac{1}{112} \left(\frac{28}{3x_4} + \frac{56x_4}{9x_5^2} + \frac{56x_4}{9x_6^2} + \frac{56x_4}{9x_7^2} \right), \\ r_5 &= \frac{1}{2x_5} + \frac{1}{9} \left(\frac{x_5}{x_6x_7} - \frac{x_6}{x_7x_5} - \frac{x_7}{x_6x_5} \right) - \frac{1}{64} \left(\frac{4x_2}{3x_5^2} + \frac{4x_3}{3x_5^2} + \frac{56x_4}{9x_5^2} \right), \\ r_6 &= \frac{1}{2x_6} + \frac{1}{9} \left(\frac{x_6}{x_7x_5} - \frac{x_7}{x_6x_5} - \frac{x_5}{x_7x_6} \right) - \frac{1}{64} \left(\frac{4x_1}{3x_6^2} + \frac{4x_3}{3x_6^2} + \frac{56x_4}{9x_6^2} \right), \\ r_7 &= \frac{1}{2x_7} + \frac{1}{9} \left(\frac{x_7}{x_5x_6} - \frac{x_6}{x_7x_5} - \frac{x_5}{x_7x_6} \right) - \frac{1}{64} \left(\frac{4x_1}{3x_7^2} + \frac{4x_2}{3x_7^2} + \frac{56x_4}{9x_7^2} \right). \end{aligned} \tag{4.12}$$

We consider the system of equations given by

$$r_1 - r_2 = 0, r_2 - r_3 = 0, r_3 - r_4 = 0, r_4 - r_5 = 0, r_5 - r_6 = 0, r_6 - r_7 = 0. \tag{4.13}$$

Since there are 7 variables in the system of equations, which is quite complicated for computer to calculate the Gröbner bases, we normalize them by $x_6 = x_7 = 1$, then it is easy to find that $x_2 = x_3$ from the

equations in (4.12), as a result, we have the following system of equations:

$$\begin{aligned}
g_1 &= 8x_1^2x_3x_5^2 - 4x_1x_3^2x_5^2 - 4x_1x_3^2 - x_1x_5^2 + x_3x_5^2 = 0, \\
g_2 &= 4x_3^2x_4x_5^2 - 4x_3x_4^2x_5^2 + 4x_3^2x_4 - 2x_3x_4^2 - 3x_3x_5^2 + x_4x_5^2 = 0, \\
g_3 &= 16x_4^2x_5^2 - 16x_4x_5^3 + 6x_3x_4 + 22x_4^2 - 40x_5x_4 + 12x_5^2 = 0, \\
g_4 &= 3x_1x_5^2 + 3x_3x_5^2 + 14x_4x_5^2 + 32x_5^3 - 72x_5^2 - 6x_3 - 14x_4 + 40x_5 = 0.
\end{aligned} \tag{4.14}$$

We consider a polynomial ring $R = Q[z, x_1, x_3, x_4, x_5]$ and an ideal I generated by $\{g_1, g_2, g_3, g_4, zx_1x_3x_4x_5 - 1\}$ to find non-zero solutions of equations (4.14). We take a lexicographic order $>$ with $z > x_1 > x_3 > x_4 > x_5$ for a monomial ordering on R . Then with the aid of computer, the following polynomial is contained in the Gröbner basis for the ideal I

$$(x_5 - 1)(4949x_5^3 - 9379x_5^2 + 5155x_5 - 875) \cdot h(x_5), \tag{4.15}$$

where $h(x_5)$ is a polynomial of degree 25 given by

$$\begin{aligned}
h(x_5) &= 25101347481190400x_5^{25} - 213612622522613760x_5^{24} + 1125174177049870336x_5^{23} \\
&\quad - 4398212675755048960x_5^{22} + 13830794079039782912x_5^{21} - 36611831495905378304x_5^{20} \\
&\quad + 83642128611649716224x_5^{19} - 167796138043083587584x_5^{18} + 299027316357649125376x_5^{17} \\
&\quad - 477090509137235365888x_5^{16} + 685232950401086713856x_5^{15} - 888981413909110722560x_5^{14} \\
&\quad + 1043617887134219845504x_5^{13} - 1109092082064780894976x_5^{12} + 1065873428902655206688x_5^{11} \\
&\quad - 924019385502926205728x_5^{10} + 719633332172147554621x_5^9 - 500367766771546562545x_5^8 \\
&\quad + 307906916846444099368x_5^7 - 165616036629637255130x_5^6 + 76466724878453429510x_5^5 \\
&\quad - 29517565522401301760x_5^4 + 9135633690673393900x_5^3 - 2105338765392512650x_5^2 \\
&\quad + 314822238961211625x_5 - 22360268064771875
\end{aligned}$$

In equation 4.15, we can get four solutions, namely 1, 0.3741245714, 0.4352557643, 1.085749994.

For $x_5 = 1$, we have $x_5 = x_6 = x_7 = 1$, then by Proposition 4.3, we know the left-invariant Einstein metrics corresponding to these solutions are all naturally reductive.

For other three solutions, the corresponding solutions of the system of equations $\{g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0\}$ with $x_1x_3x_4x_5 \neq 0$ are as follows:

$$\begin{aligned}
&\{x_1 = 0.06992197765, x_3 = x_4 = x_5 = 0.3741245714\}, \\
&\{x_1 = 1.574157664, x_3 = x_4 = x_5 = 0.4352557643\}, \\
&\{x_1 = 0.1259345024, x_3 = x_4 = x_5 = 1.085749994\}.
\end{aligned}$$

Due to Proposition 4.3, the left-invariant Einstein metrics induced by these solutions are all naturally reductive.

The solutions of $h(x_5) = 0$ can be given approximately by $\{x_5 = 0.3952383758, x_5 = 0.4800791989, x_5 = 0.4889224428, x_5 = 0.6243909850, x_5 = 0.8764616162, x_5 = 0.9877146527, x_5 = 1.214528817\}$ and the corresponding solutions of the system of equations $\{g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0, h(x_5) = 0\}$ with $x_1 x_3 x_4 x_5 \neq 0$ are as follows:

$$\{x_1 = 0.07185376030, x_3 = 0.08311707788, x_4 = 0.5173806186, x_5 = 0.3952383758\},$$

$$\{x_1 = 1.505180802, x_3 = 0.09335085020, x_4 = 0.6214188681, x_5 = 0.4800791989\},$$

$$\{x_1 = 0.07131646202, x_3 = 0.6268846419, x_4 = 0.2651770624, x_5 = 0.4889224428\},$$

$$\{x_1 = 1.506498452, x_3 = 0.8045481479, x_4 = 0.3009635903, x_5 = 0.6243909850\},$$

$$\{x_1 = 1.119004750, x_3 = 1.1136471437, x_4 = 1.063993479, x_5 = 0.8764616162\},$$

$$\{x_1 = 0.08089954767, x_3 = 0.08095556860, x_4 = 0.2627271790, x_5 = 0.9877146527\},$$

$$\{x_1 = 0.1003478332, x_3 = 1.505404155, x_4 = 0.3341288081, x_5 = 1.214528817\},$$

Due to Proposition 4.3, the left-invariant Einstein metrics induced by these solutions are all non-naturally reductive.

In summarize, we find 7 different non-naturally reductive left-invariant Einstein metrics on E_7 -I. **Case of E_7 -II.** By Lemma 3.5, the components of Ricci tensor with respect to the metric (3.17) are as follows:

$$\begin{cases} r_0 = \frac{1}{4} \left(\frac{4u_0}{9x_3^2} + \frac{5u_0}{9x_4^2} \right), \\ r_1 = \frac{1}{12} \left(\frac{1}{3x_1} + \frac{x_1}{x_3^2} + \frac{5x_1}{3x_5^2} \right), \\ r_2 = \frac{1}{140} \left(\frac{35}{3x_2} + \frac{35x_2}{9x_3^2} + \frac{70x_2}{9x_4^2} + \frac{35x_2}{3x_5^2} \right), \\ r_3 = \frac{1}{2x_3} + \frac{5}{36} \left(\frac{x_3}{x_4x_5} - \frac{x_4}{x_5x_3} - \frac{x_5}{x_3x_4} \right) - \frac{1}{48} \left(\frac{4u_0}{9x_3^2} + \frac{x_1}{x_3^2} + \frac{35x_2}{9x_3^2} \right), \\ r_4 = \frac{1}{2x_4} + \frac{1}{9} \left(\frac{x_4}{x_5x_3} - \frac{x_5}{x_3x_4} - \frac{x_3}{x_4x_5} \right) - \frac{1}{60} \left(\frac{5u_0}{9x_4^2} + \frac{70x_2}{9x_4^2} \right), \\ r_5 = \frac{1}{2x_5} + \frac{1}{12} \left(\frac{x_5}{x_3x_4} - \frac{x_3}{x_4x_5} - \frac{x_4}{x_5x_3} \right) - \frac{1}{80} \left(\frac{5x_1}{3x_5^2} + \frac{35x_2}{3x_5^2} \right). \end{cases}$$

Then we will give a criterion to decide whether a metric of the form (3.17) is naturally reductive.

Proposition 4.4. If a left-invariant metric $\langle \cdot, \cdot \rangle$ of the form (3.17) on $G = E_7$ is naturally reductive with respect to $G \times L$ for some closed subgroup L of G , then one of the following holds:

$$1) u_0 = x_1 = x_2 = x_3, x_4 = x_5 \quad 2) u_0 = x_2 = x_4, x_3 = x_5 \quad 3) x_1 = x_2 = x_5, x_3 = x_4 \quad 4) x_3 = x_4 = x_5.$$

Conversely, if one of the conditions 1), 2), 3), 4) holds, then the metric $\langle \cdot, \cdot \rangle$ of the form (3.17) is naturally reductive with respect to $G \times L$ for some closed subgroup L of G .

Proof. Let \mathfrak{l} be the Lie algebra of L . Then we have either $\mathfrak{l} \subset \mathfrak{k}$ or $\mathfrak{l} \not\subset \mathfrak{k}$. For the case of $\mathfrak{l} \not\subset \mathfrak{k}$. Let \mathfrak{h} be the subalgebra of \mathfrak{g} generated by \mathfrak{l} and \mathfrak{k} . Since $\mathfrak{g} = \mathfrak{T} \oplus A_1 \oplus A_5 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$ and the structure

of generalized Wallach spaces, there must only one \mathfrak{p}_i contained in \mathfrak{h} . If $\mathfrak{p}_1 \subset \mathfrak{h}$, then $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}_1 \cong A_7$, which is in fact the set of fixed points of the involutive automorphism σ . Due to Theorem 2.1, we have $u_0 = x_1 = x_2 = x_3, x_4 = x_5$. If $\mathfrak{p}_2 \subset \mathfrak{h}$, then $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}_2 \cong A_1 \oplus D_6$, which is in fact the set of the fixed points of the involutive automorphism τ , as a result of Theorem 2.1, we have $u_0 = x_2 = x_4, x_3 = x_5$. If $\mathfrak{p}_3 \subset \mathfrak{h}$, then $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}_3 \cong T \oplus E_6$, which is corresponds to the involutive $\sigma\tau = \tau\sigma$ [12], due to Theorem 2.1, we have $x_1 = x_2 = x_5, x_3 = x_4$.

We proceed with the case $\mathfrak{l} \subset \mathfrak{k}$. Because the orthogonal complement \mathfrak{l}^\perp of \mathfrak{l} with respect to B contains the orthogonal complement \mathfrak{k}^\perp of \mathfrak{k} , it follows that $\mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3 \subset \mathfrak{l}^\perp$. Since the invariant metric $\langle \cdot, \cdot \rangle$ is naturally reductive with respect to $G \times L$, we conclude that $x_3 = x_4 = x_5$ by Theorem 2.1.

The converse is a direct conclusion of Theorem 2.1. \square

Recall that the homogeneous Einstein equation for the left-invariant metric $\langle \cdot, \cdot \rangle$ is given by

$$\{r_0 - r_1 = 0, r_1 - r_2 = 0, r_2 - r_3 = 0, r_3 - r_4 = 0, r_4 - r_5 = 0\}.$$

We normalize the metric by setting $u_0 = 1$, then the homogeneous Einstein equation is equivalent to the following system of equations:

$$\left\{ \begin{array}{l} g_0 = -5x_1^2x_3^2x_4^2 - 3x_1^2x_4^2x_5^2 - x_3^2x_4^2x_5^2 + 5x_1x_3^2x_5^2 + 4x_1x_4^2x_5^2 = 0, \\ g_1 = 5x_1^2x_2x_3^2x_4^2 + 3x_1^2x_2x_4^2x_5^2 - 3x_1x_2^2x_3^2x_4^2 - 2x_1x_2^2x_3^2x_5^2 - x_1x_2^2x_4^2x_5^2 - 3x_1x_3^2x_4^2x_5^2 \\ \quad + x_2x_3^2x_4^2x_5^2 = 0, \\ g_2 = 9x_1x_2x_4^2x_5^2 + 36x_2^2x_3^2x_4^2 + 24x_2^2x_3^2x_5^2 + 47x_2^2x_4^2x_5^2 - 60x_2x_3^3x_4x_5 + 60x_2x_3x_4^3x_5 \\ \quad - 216x_2x_3x_4^2x_5^2 + 60x_2x_3x_4x_5^3 + 36x_3^2x_4^2x_5^2 + 4x_2x_4^2x_5^2 = 0, \\ g_3 = -9x_1x_4^2x_5 + 56x_2x_3^2x_5 - 35x_2x_4^2x_5 + 108x_3^3x_4 - 216x_3^2x_4x_5 - 108x_3x_4^3 + 216x_3x_4^2x_5 \\ \quad - 12x_3x_4x_5^2 + 4x_3^2x_5 - 4x_4^2x_5 = 0, \\ g_4 = 9x_1x_3x_4^2 + 63x_2x_3x_4^2 - 56x_2x_3x_5^2 - 12x_3^2x_4x_5 - 216x_3x_4^2x_5 + 216x_3x_4x_5^2 + 84x_4^3x_5 \\ \quad - 84x_4x_5^3 - 4x_3x_5^2 = 0. \end{array} \right.$$

Consider the polynomial ring $R = \mathbb{Q}[z, x_1, x_2, x_3, x_4, x_5]$ and the ideal I , generated by polynomials $\{zx_1x_2x_3x_4x_5 - 1, g_0, g_1, g_2, g_3, g_4\}$. We take a lexicographic ordering $>$, with $z > x_1 > x_2 > x_3 > x_4 > x_5$ for a monomial ordering on R . Then, by the aid of computer, we see that a Gröbner basis for the ideal I contains a polynomial of x_5 given by

$$(x_5 - 1)(1067x_5 - 392)(2x_5 - 7)(875x_5^3 - 5155x_5^2 + 9379x_5 - 4949)h(x_5),$$

where $h(x_5)$ is a polynomial of degree 78. Since the length of this polynomial may affect the readers to read, we put it in the Appendix III.

We remark that x_1, x_2, x_3, x_4 can be written into a polynomial of x_5 with coefficient of rational numbers. By solving $h(x_5) = 0$ numerically, we get 6 solutions, namely $x_5 \approx 0.3954420465, x_5 \approx 0.7869165511, x_5 \approx 1.022441180, x_5 \approx 1.525178916, x_5 \approx 2.907605999, x_5 \approx 3.996569735$. Further more, the corresponding solutions of the system of equations $\{g_0 = 0, g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0, h(x_5) = 0\}$ with $x_1 x_2 x_3 x_4 x_5 \neq 0$ are as follows:

$$\begin{aligned} &\{x_1 \approx 0.6527831128, x_2 \approx 0.4342037927, x_3 \approx 0.7023547363, x_4 \approx 0.7181567785, x_5 \approx 0.3954420465\}, \\ &\{x_1 \approx 1.238139339, x_2 \approx 0.2406838191, x_3 \approx 0.9516792542, x_4 \approx 0.6904065038, x_5 \approx 0.7869165511\}, \\ &\{x_1 \approx 0.07716292844, x_2 \approx 0.2617256871, x_3 \approx 1.140229546, x_4 \approx 0.6923748274, x_5 \approx 1.022441180\}, \\ &\{x_1 \approx 0.1125592068, x_2 \approx 0.3602427458, x_3 \approx 0.7175220814, x_4 \approx 1.576177679, x_5 \approx 1.525178916\}, \\ &\{x_1 \approx 1.055549830, x_2 \approx 0.7580899024, x_3 \approx 0.9219014311, x_4 \approx 2.908338968, x_5 \approx 2.907605999\}, \\ &\{x_1 \approx 0.3623080653, x_2 \approx 1.354302959, x_3 \approx 1.066348568, x_4 \approx 4.005454245, x_5 \approx 3.996569735\}. \end{aligned}$$

Due to Proposition 4.4, we conclude that these six solutions induce six different non-naturally reductive left-invariant Einstein metrics on E_7 .

For $x_5 = 1, x_5 = \frac{392}{1067}$ and $x_5 = \frac{7}{2}$, the corresponding solution of the system of equations $\{g_0 = 0, g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0\}$ with $x_1 x_2 x_3 x_4 x_5 \neq 0$ are as follows:

$$\begin{aligned} &\{x_1 = x_2 = x_3 = x_4 = x_5 = 1\}, \\ &\{x_1 = x_2 = x_5 = \frac{392}{1067}, x_3 = x_4 = \frac{742}{1067}\}, \\ &\{x_1 = x_2 = x_3 = 1, x_4 = x_5 = \frac{7}{2}\}. \end{aligned}$$

Due to Proposition 4.4, the left invariant Einstein metrics induced by these three solutions are all naturally reductive.

For $(875x_5^3 - 5155x_5^2 + 9379x_5 - 4949) = 0$, the solutions of the system of equations $\{g_0 = 0, g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0, (875x_5^3 - 5155x_5^2 + 9379x_5 - 4949) = 0\}$ with $x_1 x_2 x_3 x_4 x_5 \neq 0$ are as follows:

$$\begin{aligned} &\{x_1 \approx 0.1159884900, x_2 = x_4 = 1, x_3 = x_5 \approx 0.9210223401\}, \\ &\{x_1 \approx 3.616626805, x_2 = x_4 = 1, x_3 = x_5 \approx 2.297499727\}, \\ &\{x_1 \approx 0.1868949087, x_2 = x_4 = 1, x_3 = x_5 \approx 2.672906503\}. \end{aligned}$$

Due to Proposition 4.4, the left invariant Einstein metrics induced by these three solutions are all naturally reductive.

In conclusion, we find six different left-invariant Einstein metrics on E_7 which are non-naturally reductive.

Case of E_6 -II. By Lemma 3.6, the components of Ricci tensor with respect to the metric (3.27) can be expressed as follows:

$$\left\{ \begin{array}{l} r_0 = \frac{1}{4} \left(\frac{u_0}{2x_4^2} + \frac{u_0}{2x_5^2} \right), \\ r_1 = \frac{1}{12} \left(\frac{1}{2x_1} + \frac{x_1}{x_5^2} + \frac{3x_1}{2x_6^2} \right), \\ r_2 = \frac{1}{12} \left(\frac{1}{2x_2} + \frac{x_2}{x_4^2} + \frac{3x_2}{2x_6^2} \right), \\ r_3 = \frac{1}{60} \left(\frac{5}{x_3} + \frac{5x_3}{2x_4^2} + \frac{5x_3}{2x_5^2} + \frac{5x_3}{x_6^2} \right), \\ r_4 = \frac{1}{2x_4} + \frac{1}{8} \left(\frac{x_4}{x_5x_6} - \frac{x_5}{x_6x_4} - \frac{x_6}{x_4x_5} \right) - \frac{1}{32} \left(\frac{u_0}{2x_4^2} + \frac{x_2}{x_4^2} + \frac{5x_3}{2x_4^2} \right), \\ r_5 = \frac{1}{2x_5} + \frac{1}{8} \left(\frac{x_5}{x_6x_4} - \frac{x_6}{x_4x_5} - \frac{x_4}{x_5x_6} \right) - \frac{1}{32} \left(\frac{u_0}{2x_5^2} + \frac{x_1}{x_5^2} + \frac{5x_3}{2x_5^2} \right), \\ r_6 = \frac{1}{2x_6} + \frac{1}{12} \left(\frac{x_6}{x_4x_5} - \frac{x_4}{x_5x_6} - \frac{x_5}{x_6x_4} \right) - \frac{1}{48} \left(\frac{3x_1}{2x_6^2} + \frac{3x_2}{2x_6^2} + \frac{5x_3}{x_6^2} \right). \end{array} \right.$$

The we will give a criterion to decide whether a left-invariant metric of the form (3.27) on E_6 is naturally reductive.

Proposition 4.5. If a left-invariant metric $\langle \cdot, \cdot \rangle$ of the form (3.27) on $G = E_6$ is naturally reductive with respect to $G \times L$ for some closed subgroup L of G , then one of the following holds:

- 1) $u_0 = x_2 = x_3 = x_4, x_5 = x_6$ 2) $u_0 = x_1 = x_3 = x_5, x_4 = x_6$ 3) $x_1 = x_2 = x_3 = x_6, x_4 = x_5$ 4) $x_4 = x_5 = x_6$.

Conversely, if one of the conditions 1), 2), 3), 4) holds, then the metric $\langle \cdot, \cdot \rangle$ of the form (3.27) is naturally reductive with respect to $G \times L$ for some closed subgroup L of G .

Proof. Let \mathfrak{l} be the Lie algebra of L . Then we have either $\mathfrak{l} \subset \mathfrak{k}$ or $\mathfrak{l} \not\subset \mathfrak{k}$. For the case of $\mathfrak{l} \not\subset \mathfrak{k}$. Let \mathfrak{h} be the subalgebra of \mathfrak{g} generated by \mathfrak{l} and \mathfrak{k} . Since $\mathfrak{g} = \mathbb{T} \oplus A_1^1 \oplus A_1^2 \oplus A_3 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$ and the structure of generalized Wallach spaces, there must only one \mathfrak{p}_i contained in \mathfrak{h} . If $\mathfrak{p}_1 \subset \mathfrak{h}$, then $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}_1 \cong A_1^1 \oplus A_5$, which is in fact the set of fixed points of the involutive automorphism σ . Due to Theorem 2.1, we have $u_0 = x_2 = x_3 = x_4, x_5 = x_6$. If $\mathfrak{p}_2 \subset \mathfrak{h}$, then $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}_2 \cong A_1^2 \oplus A_5$, which is in fact the set of the fixed points of the involutive automorphism τ , as a result of Theorem 2.1, we have $u_0 = x_1 = x_3 = x_5, x_4 = x_6$. If $\mathfrak{p}_3 \subset \mathfrak{h}$, then $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}_3 \cong \mathbb{T} \oplus D_5$, which is corresponds to the involutive $\sigma\tau = \tau\sigma$ [12], due to Theorem 2.1, we have $x_1 = x_2 = x_3 = x_6, x_4 = x_5$.

We proceed with the case $\mathfrak{l} \subset \mathfrak{k}$. Because the orthogonal complement \mathfrak{l}^\perp of \mathfrak{l} with respect to B contains the orthogonal complement \mathfrak{k}^\perp of \mathfrak{k} , it follows that $\mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3 \subset \mathfrak{l}^\perp$. Since the invariant metric $\langle \cdot, \cdot \rangle$ is naturally reductive with respect to $G \times L$, we conclude that $x_4 = x_5 = x_6$ by Theorem 2.1.

The converse is a direct conclusion of Theorem 2.1. \square

Recall that the homogeneous Einstein equation for the left-invariant metric $\langle \cdot, \cdot \rangle$ is given by

$$\{r_0 - r_1 = 0, r_1 - r_2 = 0, r_2 - r_3 = 0, r_3 - r_4 = 0, r_4 - r_5 = 0\}.$$

Then finding Einstein metrics of the form (3.27) reduces to finding the positive solutions of the above system and we normalize the metric by setting $x_6 = 1$, then the homogeneous Einstein equation is equivalent to the following system of equations:

$$\left\{ \begin{array}{l} g_0 = -3x_1^2x_4^2x_5^2 + 3u_0x_1x_4^2 + 3u_0x_1x_5^2 - 2x_1^2x_4^2 - x_4^2x_5^2 = 0, \\ g_1 = 3x_1^2x_2x_4^2x_5^2 - 3x_1x_2^2x_4^2x_5^2 + 2x_1^2x_2x_4^2 - 2x_1x_2^2x_5^2 - x_1x_4^2x_5^2 + x_2x_4^2x_5^2 = 0, \\ g_2 = 3x_2^2x_3x_4^2x_5^2 - 2x_2x_3^2x_4^2x_5^2 + 2x_2^2x_3x_5^2 - x_2x_3^2x_4^2 - x_2x_3^2x_5^2 - 2x_2x_4^2x_5^2 + x_3x_4^2x_5^2 = 0, \\ g_3 = 16x_3^2x_4^2x_5^2 - 24x_3x_4^3x_5 + 24x_3x_4x_5^3 + 3u_0x_3x_5^2 + 6x_2x_3x_5^2 + 8x_3^2x_4^2 + 23x_3^2x_5^2 \\ \quad - 96x_3x_4x_5^2 + 16x_4^2x_5^2 + 24x_3x_4x_5 = 0, \\ g_4 = 16x_4^3x_5 - 16x_4x_5^3 + u_0x_4^2 - u_0x_5^2 + 2x_1x_4^2 - 2x_2x_5^2 + 5x_3x_4^2 - 5x_3x_5^2 - 32x_4^2x_5 \\ \quad + 32x_4x_5^2 = 0, \\ g_5 = 6x_1x_4x_5^2 + 6x_2x_4x_5^2 + 20x_3x_4x_5^2 - 8x_4^2x_5 - 96x_4x_5^2 + 40x_5^3 - 3u_0x_4 - 6x_1x_4 - 15x_3x_4 \\ \quad + 96x_4x_5 - 40x_5 = 0. \end{array} \right.$$

Consider the polynomial ring $R = \mathbb{Q}[z, u_0, x_1, x_2, x_3, x_4, x_5]$ and the ideal I , generated by polynomials $\{zu_0x_1x_2x_3x_4x_5 - 1, g_0, g_1, g_2, g_3, g_4, g_5\}$. We take a lexicographic ordering $>$, with $z > u_0 > x_1 > x_2 > x_3 > x_4 > x_5$ for a monomial ordering on R . Then, by the aid of computer, we see that a Gröbner basis for the ideal I contains a polynomial of x_5 given by $(x_5 - 1)(17x_5 - 31)(319x_5^3 - 585x_5^2 + 298x_5 - 46) \cdot h(x_5)$, where $h(x_5)$ is a polynomial of degree 178, we put it in the Appendix IV, since its length may affect our readers to read. In fact, we remark that with the polynomials in the Gröbner basis, x_i can be written into a expression of x_{i+1}, \dots, x_5 and $i = 1, \dots, 5$, while u_0 can be written into a expression of x_1, x_2, x_3, x_4, x_5 .

By solving $h(x_5) = 0$ numerically, there exists 14 positive solutions which can be given approximately by $x_5 \approx 0.3190072071$, $x_5 \approx 0.3565775930$, $x_5 \approx 0.4054489785$, $x_5 \approx 0.4709163886$, $x_5 \approx 0.5455899299$, $x_5 \approx 0.7832400305$, $x_5 \approx 1.000773211$, $x_5 \approx 1.002658584$, $x_5 \approx 1.003465783$, $x_5 \approx 1.006528315$, $x_5 \approx 1.069488872$, $x_5 \approx 1.155548556$, $x_5 \approx 1.646506483$, $x_5 \approx 1.695781258$. Moreover, the corresponding solutions of the system of equations $\{g_0 = 0, g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0, g_5 = 0, h(x_5) = 0\}$ with $u_0x_1x_2x_3x_4x_5 \neq 0$ can be split into the following 7 groups:

1. $\left\{ \begin{array}{l} \{u_0 \approx 0.3120392058, x_1 \approx 0.1471819373, x_2 \approx 0.1040632043, x_3 \approx 0.4015791280, x_4 \approx 1.003465783, x_5 \approx 0.3190072071\}, \\ \{u_0 \approx 0.3120392058, x_1 \approx 0.1040632043, x_2 \approx 0.1471819373, x_3 \approx 0.4015791280, x_4 \approx 0.3190072071, x_5 \approx 1.003465783\}, \end{array} \right.$
2. $\left\{ \begin{array}{l} \{u_0 \approx 0.3832451893, x_1 \approx 0.4156187592, x_2 \approx 0.1033703333, x_3 \approx 0.2795983424, x_4 \approx 1.000773211, x_5 \approx 0.3565775930\}, \\ \{u_0 \approx 0.3832451893, x_1 \approx 0.1033703333, x_2 \approx 0.4156187592, x_3 \approx 0.2795983424, x_4 \approx 0.3565775930, x_5 \approx 1.000773211\}, \end{array} \right.$
3. $\left\{ \begin{array}{l} \{u_0 \approx 0.4028095641, x_1 \approx 0.1658986529, x_2 \approx 1.591316257, x_3 \approx 0.5073650261, x_4 \approx 1.006528315, x_5 \approx 0.4054489785\}, \\ \{u_0 \approx 0.4028095641, x_1 \approx 1.591316257, x_2 \approx 0.1658986529, x_3 \approx 0.5073650261, x_4 \approx 0.4054489785, x_5 \approx 1.006528315\}, \end{array} \right.$

[illegible]

$$\begin{aligned}
& 10057112312755464433983066058320195619101802597197233470570592201692916613350578305131786195545412768 x_6^{55} + \\
& 7815426696830648061914604250266266973243034097843367116915760277296648396423251065688813292827296958 x_6^{54} - \\
& 5959041500741504739342676230879226001046706924977493457884142602249121692721907973267744556413515708 x_6^{53} + \\
& 4456465144171912344654772807258183612540227615811733523998081212234586349153413960140835167900125575 x_6^{52} - \\
& 3267582019130579458571931040636411368414210107466090031294076856865039471544160629816891300649138914 x_6^{51} + \\
& 2348019298622537219931845106040325224449376841127936407087717058427605648457057613218067167306832799 x_6^{50} - \\
& 1652797965624984944364734501531548328098700190410203729083639848818702551406718676413560442288648794 x_6^{49} + \\
& 1139126533749155188323113784381862130506344272474638522684929310780995606920131814752352645015571154 x_6^{48} - \\
& 768312629130087932486704909567133368795476393558489294068272136432855429858675357821521262760285172 x_6^{47} + \\
& 506859330039629945929036123576903762986470201996591911552482105191542415407606632542148413760601644 x_6^{46} - \\
& 326874849254477840682384310229946957489771134083669968516994727481563164015454638090887344294345670 x_6^{45} + \\
& 205955910185909378300314035545530956983528846439074840056626177783231992263223908507756385078838752 x_6^{44} - \\
& 126710981824123666170968229538116813326554434163226695218742665671948434421611801101640777890740102 x_6^{43} + \\
& 76075669993045192563804968832318949836021878712164829516745508159362085161521962927802256246872538 x_6^{42} - \\
& 44546123774203858562737203316507694133062777619959630084625346029037382741728458814705943928706820 x_6^{41} + \\
& 25424079948814984629207093374323401542188106890965494590130148254438572882740924871317462129769286 x_6^{40} - \\
& 14134754167722643633726939221604262601223529929465692488570987952206549624337306012138781911925014 x_6^{39} + \\
& 7650210550322154812189001607124054641751922374805051507835273092274564010735771078329022936177926 x_6^{38} - \\
& 4028400549539187018992894114891089589580150128650696195170202803559075981096739802818599329458446 x_6^{37} + \\
& 20625046900818299000180289859594844151118352103047456363540289957813121404803502918698507464289716 x_6^{36} - \\
& 1026089059219399650760857906769747803720756098122009477430762995721873270212987522789252292364364 x_6^{35} + \\
& 495703837079474212874671053140765282382801340577813553149110407378105516432073961287160046768876 x_6^{34} - \\
& 232390733552737045335629185654999832651320240145225223444250737827560540662735451238646960136598 x_6^{33} + \\
& 105652022802101393188141512641862236798001179702090179364971313726411114144364573456998404552476 x_6^{32} - \\
& 46546959217278241385295256857636422335592436377929517788382230391252368928319967388969151737398 x_6^{31} + \\
& 19858100541803106926904408901752742130340672229494971316383032353051521167461457189672834072094 x_6^{30} - \\
& 8197455866919393826435150321412801803120761170805735702648847245471335334010157085431416016164 x_6^{29} + \\
& 3271585422148232882984609479876199064691579238439876178430988984960889820781934191606548102735 x_6^{28} - \\
& 1261233266374010693827434138105796490286236015474740910991540133982120592512441565670641934156 x_6^{27} + \\
& 469228512106326617112460403969728143919350145959862950061782396864465244470750494493757108597 x_6^{26} - \\
& 168301715723186009564910019276060906015115581793854255120274409266120834901127339810646645928 x_6^{25} + \\
& 5813477606164058489879938134874407675849646364688326565254499838031295975064452921070505454 x_6^{24} - \\
& 19315871932373280332127853706359356286529593585317343480616017086638110758962679878721592032 x_6^{23} + \\
& 6165442090907758467003182162168410728084866700934563866146171038163427792394988732447742576 x_6^{22} - \\
& 1887871491252749071165094213916356664798522111340622135482992091045000367356910998396504640 x_6^{21} + \\
& 553685436076448230582095063647153018919049767435862585449491653881668083955425859890407200 x_6^{20} - \\
& 155270069022930306455064658339299828485707651572993944747264551460206031752289211982987200 x_6^{19} + \\
& 41554020009974686461242824300388850412981599048711760342154161186385016909464516477310800 x_6^{18} - \\
& 10590245914774140828048927818702238404535327456947357218449153450593999656631562693776000 x_6^{17} + \\
& 2563987767983678891215691704653502301548996316078479440915761179635744715385150414860000 x_6^{16} - \\
& 588100297307408593160912180295955225449377126353066214406197496763592683855730919488000 x_6^{15} + \\
& 127394819920587345997715014771148502743677895787382759422010047188016607123337846352000 x_6^{14} - \\
& 25968706971106515156435215854797406123655066708180261542604377597361944349585098624000 x_6^{13} + \\
& 4960507940027602159108171414133540542870357682494160638808628474544074120268857600000 x_6^{12} - \\
& 883568435934032491854600142821810738907025437706278188778615895234588463202734080000 x_6^{11} + \\
& 145897682005731748694196345430889511740879526344443757233716282574010926547568640000 x_6^{10} - \\
& 22175866222857978558060838480620319985670100890081201714818580060768067723673600000 x_6^9 + \\
& 3075862118928405963111852344998231570651617836342389807394973285218950438502400000 x_6^8 - \\
& 385105259410167229527440401287446716975632740993559048809336040524404994867200000 x_6^7 + \\
& 42917339672778567692638781664562933815548814423377227738175562693341224960000000 x_6^6 - \\
& 4178590536688350662228868634718089493254270404833603948496989437288710144000000 x_6^5 + \\
& 346352374181513467040566508305589632427483211407599391709775625992142848000000 x_6^4 - \\
& 23523257372893535305090108878056327495383233042527291749486130479759360000000 x_6^3 + \\
& 1230919766784379333820283776146657594954326996169558341670191811788800000000 x_6^2 -
\end{aligned}$$

4425834669660109969214783489482241934346982554062710942618091520000000000 x_6 +
8227594418459832935555146743182600186707464628050649891733504000000000

6. APPENDIX II

We put $h(x_6)$ here as follows:

$h(x_6) = 4483391625806314902278623053770051604405925444922144197732882524176570461483849883534229504 x_6^{114}$
 $- 205326408642245030427921175421329504773285549623799716231703635104414332571732690022581665792 x_6^{113}$
 $+ 474822734328891299029688534396637345706993828346496008626639317277439376636833307469432356864 x_6^{112}$
 $- 73857027765851476864907995448674285385990573072187964007497919962287095629599881653594965934080 x_6^{111}$
 $+ 867664040149060106328280687969035749162428713361957811145995338185100965861669204561179292205056 x_6^{110}$
 $- 8190861944847112540343332023107832167424206756712692909839631199237350246201450512424812054315008 x_6^{109}$
 $+ 6452420142081238765564839477345079436837813397868740061962179983314070957918412138538943926763520 x_6^{108}$
 $- 43471282282437200305047238826737804111974281774961322415630966710723431930513864104548900129472512 x_6^{107}$
 $+ 2545970008909317278697022351939013486712452518283817781567560897727987929782362553710098265197772800 x_6^{106}$
 $- 13097028510927733556711072703012531491895193895358908087070737594767573219855012966233119733042905088 x_6^{105}$
 $+ 59489010993776772645935266065220062725433224359410810434669261324137366389534849780109565418018963456 x_6^{104}$
 $- 238513405926928679034448569686833432897697263280851985483727297580965902485083469530724927053045432320 x_6^{103}$
 $+ 837192767901400659327388305673834160162972688262551443375142899312686418524934321672790512987409481728 x_6^{102}$
 $- 25130991367084329367429340494973102960034191985334808829215935856838896888677689779264842174440669184 x_6^{101}$
 $+ 6053719990602192749861843286183348238861603167575816936543214471122284892335479278172815542584764006040 x_6^{100}$
 $- 916387631164299105539598749631150596871330962512532720142436345754112557269606001994254021795015294976 x_6^{99}$
 $- 9024249443985225373330180204805081933528916042035915597129425510528112051248275814341201244956282322944 x_6^{98}$
 $+ 14207493421789601577153986432050136617159306766732050296183577155353072731641661116166010165465113457328128 x_6^{97}$
 $- 70584199968738344877345292965675225055162592946169846889123331088101472557183033537346557215306246258688 x_6^{96}$
 $+ 2556963252800373545141667373003943578430092496177297523595625717533131530755386426887189567118496556384256 x_6^{95}$
 $- 7506000937205795890819942467284655011933626507544074468697398990596153361536969409902109321077199164407808 x_6^{94}$
 $+ 17936874296373676114196887995833491364108238894730113646733427249384695776872725472056415982690935691542528 x_6^{93}$
 $- 31994491935711999747347405170149967952484298573433497955772482422845439648875035140492738459008304894967808 x_6^{92}$
 $+ 23577072416713937683736594033035793565791544287767289977647307789915294750305008817333193966475822808170496 x_6^{91}$
 $+ 118049483336789073704806907341232718697539837973168098200793109647025686330009533187642821903486004605485056 x_6^{90}$
 $- 734226593700496983659859704822162571968318879856660029116807065471505272263760698526703379649368421159665664 x_6^{89}$
 $+ 267610076781176135815943894128073225826679394686887068007192972643646754163029477957216740488747100107964416 x_6^{88}$
 $- 7662993760316617988424167386186942948965267517817521495920041905265592014721029928346467644366995617925300224 x_6^{87}$
 $+ 18228878401501112954518648203563585882625088337782727970218694078395658528409378636434793400122030758336860666 x_6^{86}$
 $- 35497917132999416914255497739983952213960106447786351129608800669158645578268238705407671446370670634791337984 x_6^{85}$
 $+ 5008060407320406683100660759164681960883995393061413176438435353761210999449666445853487954362644268536823808 x_6^{84}$
 $- 157202856355778053364329805158241452724031542073857594702537987971294959895284267203702419415647310137933168640 x_6^{83}$
 $- 21325392215530130643456256092540488697583733651168480394455401620825501728764249043670442440810363342977761280 x_6^{82}$
 $+ 10074930434558661765335032242929997441841621667894278440117798220505347124355675937747780755933404183108260986880 x_6^{81}$
 $- 317104186049760352808595547606604744997187422941016394874266237599677292995197145911648361293608523220490649600 x_6^{80}$
 $+ 8192737511252059459769691589000259072047018568076663448429160778782067226324093815029594653616870160763925299200 x_6^{79}$
 $- 18305135177471481148456590646885825040465716298190833048619372186905924029582005984292092620693283163176802173600 x_6^{78}$
 $+ 356241738773517291774443717472911954467736246941837122748857500852050404507310578866325771229695512146484731576320 x_6^{77}$
 $- 58582234594937660061939858792579634811160139147261998875044075344563326032022628102019721902715587786812839428096 x_6^{76}$
 $+ 71819099318483048562659053709646755971169118190910895195097395111871506597473238712641461855192748334374136578048 x_6^{75}$
 $- 2209394024358719000728734770664218930894638199195353683879238303959476123643969362831745194258873365455764979712 x_6^{74}$
 $- 233025217903294298517141287464717280418226382056033072040305034274672769945242846989255002152281094724055960911872 x_6^{73}$
 $+ 1025523298653789662058798736792859638602688572150673287005471885539852081213238769000767234977739727429349020794880 x_6^{72}$
 $- 3059734288811407594538252169644782611421356493484183109820234882083949683418122153719523495305884612856030091542528 x_6^{71}$
 $+ 772484010883412811122985253623743171066155405872104137836175939697010487564442474281477549811357135551193290637312 x_6^{70}$
 $- 17598555229915204052409275721605938532617305913212823977587619976905852836472667837758344972101004218032171429593088 x_6^{69}$
 $+ 37214168808436892955139992394629913811032837847887399991546743647308501001802312702530950997073919761611388968124416 x_6^{68}$
 $- 74164349973524924164044748487950527176785143354532194711993428539554783092197599993287548621237068947503033156239360 x_6^{67}$
 $+ 14059922506302640149879338388244543241421231293952449285214127461284194362568804729645646538437155823194867224264704 x_6^{66}$
 $- 2551349296650284484621239356516763932181925186418561869920586892467722305062880919757671116283639555716651134943232 x_6^{65}$
 $+ 445115345117851916722473459965705479745480779085200616816510631773582234441390714046969812170649847704933366696257536 x_6^{64}$
 $- 74906171903452349434393887990337123129281204178478912151770766767775583798652397656826217325621436188989758776354816 x_6^{63}$
 $+ 1219007780972759381537137388267547796560314187219082470186252125095981682570335264334237104391399268072460319401508864 x_6^{62}$
 $- 1922267833843986235022933568021913995489988232190319502655612143006018494863365132399299614081058546517837130018135296 x_6^{61}$
 $+ 2942049957640812163906274015735702461296336164943877137584275801313056626469838939156398925995441529615603165326043200 x_6^{60}$
 $- 43762439111655903549078712208713360123726135533001924790283901505748305531767721200907371667145695529192616110802304 x_6^{59}$
 $+ 6333726504703618856653502310224966798566019634504342266079892781888679281304936160549551502977632695189789118671499520 x_6^{58}$
 $- 89276750970918878047452268284539447710420632659299007553699437176610054756257097775677636313020791928669151806494976 x_6^{57}$
 $+ 12265685956882439991845820057377301977701307980221174648635325190086145361805524317517531714569142062396466085421260528 x_6^{56}$
 $- 1643695463918133919972527483898465948288269939689267056550345400873482499769823881078022458692380993578749507375568204 x_6^{55}$
 $+ 214973502584536384153972793729374792999028085718086019955263685190878611150918156539350683986117577223540558449219294051136 x_6^{54}$
 $- 27453825635407692526495522227723181436782015685051421530035594447587824862989263800564974129586632784329227564794286480 x_6^{53}$
 $+ 34250139598857372273893286430557004669085816584142659388356305799891035604666301778847885029162327546724314447540592476 x_6^{52}$
 $- 417561993274467939434304164893079866543087213016133440056503498647913956225616797530689297745778547249926023677226648 x_6^{51}$
 $+ 49763342019653154352562591967729846304052752675277957675977061260505683059276922971558559201463011741849416976582325488 x_6^{50}$
 $- 5798748746274636633533599171800237348583887935098968647341803895525756487993561779421879523681815240678851464094886056 x_6^{49}$
 $+ 660812926168557225517180615894176711201526013997889515390102519263362839587014955768679394364920959353621947428964737 x_6^{48}$
 $- 73655294673820815925288413980144343074212843219082144168473872042616657762760922761859179435971211674668985781534517500 x_6^{47}$
 $+ 80306691545910150245367112937968208511672761873112402793431287741215694385854056693545710303318473848966530575545751580 x_6^{46}$
 $- 856530938623498275655803935149879614718012292127448490876587618595025473333644309850362532742862630521792781327819145020 x_6^{45}$
 $+ 89367533191496865457171836824211611038413396796619272675309994300077047958693420361198462461574043052964279954786643780 x_6^{44}$
 $- 9121044925753934247002472035018984390462182691618369432805884278529261913185953934119479451380811177892501648130466020 x_6^{43}$
 $+ 9105445246155996855357742213192997670275564845348405630776309072963233095896146453409671919815167821476644134852675444 x_6^{42}$
 $- 88898437541174711715857750586848730886368844816411220259916897384768072942558579930738004664931842901116211216035133972 x_6^{41}$
 $+ 84868888741265046378290537973934804950595327870250375948494626877567996926661092217987732177349541474335293535252740726 x_6^{40}$

$-7920811536911476699008917567487255920929411333818456830757744706375155818086412275304344000524148250370963873485090316 \times_6^{39}$
 $+72250776886153494923569109186657985156939727734775421593425799130418669640659884616431522278010448820239559548579599460 \times_6^{38}$
 $-6439182998046787014475286033633970761521582782708142703890132665274904763149563765091447902175939632575049165759240844 \times_6^{37}$
 $+56050056059301403897498004796897349226648237681855325925861150312204486441739258979455207815222504548273765606248331496 \times_6^{36}$
 $-476317795117877965719696340194494164557251657997779797910648523804776760453638349613177170209221510076284992362837866884 \times_6^{35}$
 $+3949908061973949157927073599405766205548262015368623778229693050332744389406034558016435258661721158636673516939597564 \times_6^{34}$
 $-31945859788755370092456761074054472831651732972799157275898200584015213478472443103682886677025951103427156238146420388 \times_6^{33}$
 $+25183722561334978336454160054965989376608115536889702639786203461738916756192687653176293304462703554397983017958570457 \times_6^{32}$
 $-19338099618373722669059669684408378980599446416023939193013170475797613589593234875606362932851846780945957933674850408 \times_6^{31}$
 $+14453573286857473064473938716154482163802115435134318539543214024100524130195352073381460755410255125893797529219895456 \times_6^{30}$
 $-10506283893683950982237258985379197783560704043783843231538653100486294357045515762738566190667231697911368567221530200 \times_6^{29}$
 $+7420676999572068797167588141139250346102101153695723803051284950741365894561917099189383422998601170039729736052998528 \times_6^{28}$
 $-508775878644159009567284077404521921647088174034583785106555486576588627972776012025436255495025920499344227899039344 \times_6^{27}$
 $+33823830755480430156844313046572281980491973564351931036448204023079706343604156193886348986611530702250237289093120 \times_6^{26}$
 $-217775881422579808271697023165625563794684467794189997304066903819557627554011437473728015218092776210095863631577936 \times_6^{25}$
 $+1356166832243486057136360018201916252457358988130700500991581357957438071388815746386879543929710277448847638477085960 \times_6^{24}$
 $-815645666020237828883820614975859631653770088733044655641730207118001267968093449599423029741642182490137553503053984 \times_6^{23}$
 $+47302144655031559194280062189676576899247172268822306993039737368139735748855607037515729677471830411607298046880173696 \times_6^{22}$
 $-264050411498286501527450861864478799401697473834016975193519596212174085659602735695340944356689801385833734708694944 \times_6^{21}$
 $+141604493847024673563277897583433233231512714531396898135844906095557682072804603943045270097096431539411007607985856 \times_6^{20}$
 $-72798151358188646984916605951368515149104134569690822065973025532312726888482172688778782109993500493509669967860224 \times_6^{19}$
 $+35791915960104934870095512437794470624977306091544210630932214416509751900942559643587069631054972130794391619136832 \times_6^{18}$
 $-16785065526227397551430648223203953785768377099815666232468321802711730642283913098885866848845514746866586876890112 \times_6^{17}$
 $+7486039386167758165498756320666538440668058449690300911364834080457175291113335212717971648404758925373300286691344 \times_6^{16}$
 $-3164690591337089269122651792673430984332571623895791846860594990256707408956006069818488869855195323713547291274240 \times_6^{15}$
 $+1263371862412637336118077448284129277800934557049604208353286016322956282676072492375524618321518497537179081405440 \times_6^{14}$
 $-4742371744598656736229065903008236257362680808571814423376106553991218384112852650618724470295923203216980460800 \times_6^{13}$
 $+166567588485401531601351575658375428881889405229134541528417132083554220056821494460709982665900520238239575909760 \times_6^{12}$
 $-54429586659678273150275580156007449410860423905145212704433824330499298121721064388192700925034251046304478330880 \times_6^{11}$
 $+164369422471440866903155293897008847106227906258231976343884514436960544901057737307644527864360164773036778127360 \times_6^{10}$
 $-455055990311633527219033435667806756730033039887054893274986715736952210291974416159798729836648828774753239040 \times_6^9$
 $+1143794330365551469610644486847760213393907293445506682022696752757580949711190716433894667610583913035584634880 \times_6^8$
 $-25789240611548325082014969056860495165606582965854940445337818055862397192510377953458445170103335234043904000 \times_6^7$
 $+51366626789697031994971660978007265502691340725200934436341529069242803083658647745944596999843687882240819200 \times_6^6$
 $-885778533824662183640035325209323345883967696847760213393907293445506682022696752757580949711190716433894667610583913035584634880 \times_6^5$
 $+1286350853431778836288662240252507434753495656746971709685224196828127797048648651729278000729076056824217600 \times_6^4$
 $-151108722890280446852576993040398716743469878173371315868991303511532560180920169534954354971901547669094400 \times_6^3$
 $+13468295516322163243823865461588698750671720482668895252627491145464137024105265628046711165122125391462400 \times_6^2$
 $-80995593679404419990999198050851090431254722130327121234710273131829159152130816806933719380294578995200 \times_6$
 $+2465279547872568705193226321305604963662010504130505462968280403875490149216360605689272167309928038400$

7. APPENDIX III

We put $h(x_5)$ here as follows:

$128347343962975370412693426749009849339839357016801236766964799153788520545469283287700441036952900345128917401600000000 \times_5^{178} -$
 $5530540078084901614541347163272585556710461227261360504342982180428179351592239804543551577526948963951376020275200000000 \times_5^{177} +$
 $1214368671181023056340825625079201542939242778107087349459445314331171252355905862118372161779163451531290451629834240000 \times_5^{176} -$
 $00 \times_5^{175} -$
 $1812431994310165286339190290550274741253258982737776277937193581998625964287755164528412074760604334773217549337021644800 \times_5^{174} +$
 $2068707462078478021064883505615211507123991283159373507839725373497169580163620572530599699435828318654118957649735188480 \times_5^{173} -$
 $0000 \times_5^{172} -$
 $1925873300646058612284297034856109247401795114532998418465537834769953774453717368954658193957404620362604996521357381468 \times_5^{171} +$
 $1522757295376626038741802388644779881996787547117412902238876564126995898728424975593066546705697211558618302730037040565 \times_5^{170} -$
 $452800 \times_5^{169} -$
 $1051322757936520155495220079568528552005553685193943817063614397077959353964356464752768181592436787886207528323855554564 \times_5^{168} +$
 $8496640 \times_5^{167} +$
 $6466180208832624164623954167835744881120569147912311659285213246326350815257644212205523180135164250850094582806452137537 \times_5^{166} -$
 $9881984 \times_5^{165} -$
 $3596848876442193547475956779805286385701797301462769959601207412927544020389729830738737743522457707905475592562932467680 \times_5^{164} +$
 $57942016 \times_5^{163} -$
 $183083054145209175207324467321553526739361141355614791684699954596003571137359633047908024378665453343164760172044776515 \times_5^{162} -$
 $267723264 \times_5^{161} -$
 $8607332481708493271697961883983485815650906558642787236387410082195867547956110322590237108086473316084344853203251259425 \times_5^{160} +$
 $888927744 \times_5^{159} -$
 $3765785531756162462427350264287410124363832376273812417161365892528408491688570163592369263093746241632193297632948022624 \times_5^{158} -$
 $0175538176 \times_5^{157} -$
 $1542758760588866064080912185086604663134560725127199843177604916390452553607595186339595386002418176472602234569399028917 \times_5^{156} +$
 $30183127040 \times_5^{155} +$
 $5948861302729476122393831944548721298294439852821189927023221924120621699253980927796068501984745210247055008344663587788 \times_5^{154} -$
 $42761396224 \times_5^{153} -$
 $2168433620503978001015246311858351279579697918033759070972940543387586412850181135391452008869571808403033665890101664503 \times_5^{152} +$
 $880721891328 \times_5^{151} +$
 $7499537721285873739505304287210342067791732541484579259340739857323203882008983520798041021251358693804347870758083031024 \times_5^{150} -$
 $468473413632 \times_5^{149} -$
 $246870493046977575549100387314406027416273460312995966687429927495116482035439308730930902233085159700116864761193703030 \times_5^{148} +$
 $6596946706432 \times_5^{147} -$
 $7755911459172691169162387197427066684138277328892163727843824411761749593503084147541538475456398973104702372303567199338 \times_5^{146} -$
 $1088770654208 \times_5^{145} -$
 $2331058312235712666892225448509846596174222520233966816340610963617511249198797415785868029717774184443317302451929362040 \times_5^{144} +$
 $81047705485312 \times_5^{143} -$
 $67162774998794686660401153158232638278819814773807434406849956148142762041754503108242985523991280790795627581337968578165 \times_5^{142} -$
 $36286227333120 \times_5^{141} -$
 $1858444060643361072383491574826396089107694238070005703922032639993004377971311240260904533619650739293066599774730420765 \times_5^{140} +$
 $791351805575168 \times_5^{139} -$

4946684911247295558626514887387912259913504869666357827546724104367661923235136463821908209878377733679310763952445549592
 908904016248832 x_5^{156} −
 1268362635471390146625495033502720783248432474716429078211589854255894741904562053987685615084639680167706889959649283875
 6448960051412992 x_5^{155} +
 3136829044349927529309080246078610614695759437088711569871550101091262533047682613429105463982477859241444089535123758134
 7039301509414912 x_5^{154} −
 7491223764029709356811637399377955598275358997168945003640574807572416180566257375258081450158359154489493889604479591633
 1762994599493632 x_5^{153} +
 1729321274353975460068086313299385761058347135937046518707069672519313793171224006543407273167152230907451040387888013318
 93454464611385344 x_5^{152} −
 3862448271322219100148003314179251789945179543745535319206591457036005905551041782644218932825546267455936900955711508691
 93583252475281408 x_5^{151} +
 8353697560808298624818573838240776081557945392632408597890113984450485534542780037572958427247794539389093564782532916911
 25573957343313920 x_5^{150} −
 1750875437025128396945875516893813752771485565823419734087812202563484252488690007146280268562438058613572858052007235138
 002290270874894336 x_5^{149} +
 3558717741163152898610084645653818401412789520711877110687028727558579770637872151654439202485152044014730072561485319944
 228895779548413952 x_5^{148} −
 7018883023774221747505869337866007795290076146413652181055494807400316300676735421670090898086016716547456109999916206449
 519670646700572672 x_5^{147} +
 134409032657745467626908053663895974028636202175170558247590554397919562760820616610806770468762956669208819682488628211
 2215871390889988096 x_5^{146} −
 2500357340996334350493014184080272021854591086360947102851033455325624121591770297969978787420202499919740127471441716297
 0442944448491134976 x_5^{145} +
 4520587476454335797927157969001199900667087694446069724497212679260433239135486965740591399930085092271431240280202006273
 9314054267520240640 x_5^{144} −
 7946807071999575205546372233951826638836565949636313435627902629207612818360294911794695283214829026118001658227256208134
 5749785210716211200 x_5^{143} +
 1358819274464904722904368996520264658037602452075211613568903004595164450690664725365245465720025382923226535487437168271
 01277137327336608768 x_5^{142} −
 2260730501023715221402951820117237525988685309020907058903498223984445669060339191697014056761522313232661929997845934749
 92095051616500285440 x_5^{141} +
 3660842969126465175883093795434890919005846002838643575918432039468292728514894869069401979867167460458228431903630436903
 41034489112715775488 x_5^{140} −
 57711894360699221710021632077546747141049919132051451118883569811576518841725496626317030674140166222839953247106293261686
 6722439728606024704 x_5^{139} +
 88590360337026478263637661218289704622147477120797995792259616642457384487164259362378208804951177351417626524836035815855
 3596241524859969856 x_5^{138} −
 13243593542905229239104932446222722176189832466152708436166911969003720588150326322038614993840850184087510296807151247921
 96657926110337094016 x_5^{137} +
 1928228147341053541805053277753352869464145616742040289190223888954237651154333070833464824361114582899354355893859852910
 04765281444161453376 x_5^{136} −
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 1177461943542470618131248328715269518620620160523025726235278887305587723708643261199372741470180277283775367742531764934
 934528 $x_5^{23} +$
 352459870759017596756740697434679649855872761844905021374167118499217354362152667054855737000124806799768236468068217344218
 89024 $x_5^{22} -$
 101148462211640964398258040702198264139855120333737285521450150284231522877704654697828850473172705829129258172716088307335
 82336 $x_5^{21} +$
 277845776669249670026113592719534200179051070301050586342016529351199814983917459135671441046941536923534166276679528022540
 2880 $x_5^{20} -$
 729222059942677239194861472602631970992826200048350708453908761070046160711682680923725375727555598800883196185895432265662
 464 $x_5^{19} +$
 182490369232807069021403284819016676754281859773867649736818460685067399592662641294817600630086707838402766296441916835758
 080 $x_5^{18} -$
 434447774417556252713537169449578275615658559171010875350475357432739654092695255407634470093845253375407341706207100319825

$92x_5^{17} +$
 $981312487711337377849734520545821034395980798786557655574632650995129713487289075138443547286859623905492491939765776338124$
 $8x_5^{16} -$
 $209672188120683804006152344764047079267202374011796846480825399560699264551760692952860839830270392859460630702219370640179$
 $2x_5^{15} +$
 $422314275671514886321123779568396098193816231242663172075394857732652916567647564791990662433359094935353521403228822962176$
 $x_5^{14} -$
 $79864922656123001672373003402242843914722460212594718228143218584471442458879920199517152848106944554516039278408268513280$
 $x_5^{13} +$
 $14114974194660343006125317651559136005123731524715696467962306387120914690527501028427382041548468495779517799388749496320$
 $x_5^{12} -$
 $2318631152616942863910854119763409708464510724161955816200337405320112889645914590041456718031593814708975738270882201600$
 $x_5^{11} +$
 $351711906340592001610413792726774343523694998021397689259098322027975998519200903802531516313342353703169108580342169600$
 $x_5^{10} -$
 $48881654482529098774644700465700353976861287780733065236104378937903373457008646072769666061615264294470604491325440000$
 $x_5^9 +$
 $6165199924406916311726448672009618574787780580033514548805089543085277996422090595590674195421977333160669442211840000$
 $x_5^8 -$
 $697265508611783523063822245221103038364102732880164181773707035276354526553076359731686155685276694260890153779200000$
 $x_5^7 +$
 $69639850156490923537983768960854449000882112021803741111317445152925716960736160612227466714320924661507948544000000x_5^6 -$
 $6019516033064772902099011570341461065066869308802412228392754929923286004456239967801891827762038101078179840000000x_5^5 +$
 $43797709923774095862004933098996973827136218815395628707280026388262269517157957648785965624334795512217600000000x_5^4 -$
 $25760391661695283164506462715706115713693855148545427268637078512865404994468648889469516554411358289920000000000x_5^3 +$
 $1148563712556502089867697244210114379584113909091646727528636419580863895074813100054898930060623872000000000000x_5^2 -$
 $34512293527648499324111859456722798709671062614568163486553797109943319310750330288373992112783360000000000000000x_5 +$
 $524157766562226639572627086349581115653126065201703317806174894734427971871427112480895139840000000000000000$

8. APPENDIX IV

We put $h(x_5)$ here as follows:

$1283473439629753704126934267490098493398393570168012367669647991537885205454692832877004410369529003451289174016000000$
 $00x_5^{18} -$
 $5530540078084901614541347163272585556710461227261360504342982180428179351592239804543551577526948963951376020275200000000$
 $x_5^{17} +$
 $1214368671181023056340825625079201542939242778107087349459445314331171252355905862118372161779163451531290451629832420000$
 $00x_5^{16} -$
 $1812431994310165286339190290550274741253258982737776277937193581998625964287755164528412074760604334773217549337021644800$
 $000x_5^{15} +$
 $2068707462078478021064883505615211507123991283159373507839725373497169580163620572530599699435828318654118957649735188480$
 $0000x_5^{14} -$
 $1925873300646058612284297034856109247401795114532998418465537834769953774453717368954658193957404620362604996521357381468$
 $16000x_5^{13} +$
 $1522757295376626038741802388644779881996787547117412902238876564126995898728424975593066546705697211558618302730037040565$
 $452800x_5^{12} -$
 $105132275793652015549522007956852855200553685193943817063614397077959353964356464752768181592436787886207528323855554564$
 $8496640x_5^{11} +$
 $6466180208832624164623954167835744881120569147912311659285213246326350815257644212205523180135164250850094582806452137537$
 $9881984x_5^{10} -$
 $3596848876442193547475956779805286385701797301462769959601207412927544020389729830738737743522457707905475592562932467680$
 $57942016x_5^{9} +$
 $1830830541452091752073244673215535267393611413556147916846999545960035711373596330479080243786654533431614760172044776515$
 $267723264x_5^{8} -$
 $8607332481708493271697961883983485815650906558642787236387410082195867547956110322590237108086473316084344853203251259425$
 $888927744x_5^{7} +$
 $3765785531756162462427350264287410124363832376273812417161365892528408491688570163592369263093746241632193297632948022624$
 $0175538176x_5^{6} -$
 $154275876058866064808912185086604663134560725127199843177604916390452553607595186339595386002418176472602234569399028917$
 $30183127040x_5^{5} +$
 $5948861302729476122393831944548721298294439852821189927023221924120621699253980927796068501984745210247055008344663587788$
 $42761396224x_5^{4} -$
 $2168433620503978001015246311858351279579697918033759070972940543387586412850181135391452008869571808403033665890101664503$
 $880721891328x_5^{3} +$
 $7499537721285873739505304287210342067791732541484579259340739857323203882008983520798041021251358693804347870758083031024$
 $468473413632x_5^{2} -$
 $2468704930469775775549100387314406027416273460312995966687429927495116482035439308730930902233085159700116864761193703030$
 $6596946706432x_5^{1} +$
 $7755911459172691169162387197427066684138277328892163727843824411761749593503084147541538475456398973104702372303567199338$
 $1088770654208x_5^{0} -$
 $2331058312235712666892225448509846596174222520233966816340610963617511249198797415785868029717774184443317302451929362040$
 $81047705485312x_5^{159} +$
 $6716277499879468660401153158232638278819814773807434406849956148142762041754503108242985523991280790795627581337968578165$
 $36286227333120x_5^{158} -$
 $1858444060643361072383491574826396089107694238070005703922032639993004377971311240260904533619650739293066599774730420765$
 $791351805575168x_5^{157} +$
 $494668491124729555862651488738791225991350486966635782754672410436766192323513646382190820987837773367931076395244549592$
 $908904016248832x_5^{156} -$
 $126836263547139014662549503350270783248432474716429078211589854255894741904562053987685615084639680167706889959649283875$
 $6448960051412992x_5^{155} +$
 $3136829044349927529309080246078610614695759437088711569871550101091262533047682613429105463982477859241444089535123758134$
 $7039301509414912x_5^{154} -$
 $7491223764029709356811637399377955598275358997168945003640574807572416180566257375258081450158359154489493889604479591633$
 $1762994599493632x_5^{153} +$
 $1729321274353975460068086313299385761058347135937046518707069672519313793171224006543407273167152230907451040387888013318$
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 $93583252475281408x_5^{151} +$
 $8353697560808298624818573838240776081557945392632408597890113984450485534542780037572958427247794539389093564782532916911$
 $25573957343313920x_5^{150} -$
 $1750875437025128396945875516893813752771485565823419734087812202563484252488690007146280268562438058613572858052007235138$

002290270874894336 $x_5^{149} +$
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 228895779548413952 $x_5^{148} -$
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 519670646700572672 $x_5^{147} +$
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 2215871390889988096 $x_5^{146} -$
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 0442944448491134976 $x_5^{145} +$
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 9314054267520240640 $x_5^{144} -$
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 5749785210716211200 $x_5^{143} +$
 1358819274464904722904368996520264658037602452075211613568903004595164450690664725365245465720025382923226535487437168271
 01277137327336608768 $x_5^{142} -$
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 41034489112715775488 $x_5^{140} -$
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 6722439728606024704 $x_5^{139} +$
 88590360337026478263637661218289704622147477120797995792259616642457384487164259362378208804951177351417626524836035815855
 359624152485996856 $x_5^{138} -$
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 96657926110337094016 $x_5^{137} +$
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 27343184778566301245697638082020449115287839566295218491265463524964789763744342140049519481489031047423991498340090089037
 3822039047022017280 $x_5^{135} +$
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 66470826431264506725268252441958349989067586344273044684172374564969673914111921389593200972749451069705616951995554397162
 21552032257127773000 $x_5^{132} -$
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 298441867693259250546 $x_5^{129} +$
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 247164325419697733428 $x_5^{128} -$
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 79145129817542588416 $x_5^{119} +$
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 38414077492176951172 $x_5^{107} +$
 76342008137265855352457154173381913204276007231457903654058261392175970556076497411261325380320027940810270982640154787009
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 71792449556531971953932454872195603675644492773984566939341763623444370920314174139255900502559173080009628439424070427939
 05460470523437518140 $x_5^{104} -$
 8902774835349314396804490714159660281266357543372726473159497890050299318071488925474014828449673921704912480683055317484
 90006229283334704360 $x_5^{103} +$

95390674243254028951416387101583884086507501671268109816504623055013937666607034381708557029350366091476713277207949663517
 75487499094709589528 x_5^{102} -
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 02097572116070256360 x_5^{101} +
 7983177547393387233398507707238880663607776647088597506503954257562846903175308829998405072345711476181455288410415947100
 6892122375577529952 x_5^{100} -
 62259913093396359529440393894415639279058410216823142055147288941120570247242267720798257239989029000507597202679444889292
 84093955133745238552 x_5^{99} +
 41789821240458939607128369026605760499680705283128064529550596100332375129726584250761953092423420196295474837565550618296
 69044944383370612043 x_5^{98} -
 21117226671067749526925361477915031392700848141975664954737679906141142109136681558694635419241480249635576154592696844169
 41442053607155074002 x_5^{97} +
 25218654402723218219412784265723850953865720122335202548657396853917596689033709014977411645525251066039904060353323050936
 5241977745737342108 x_5^{96} +
 12351901037265407518282511061924740081620887845329673297102975249899282902269201828534732039482980880851177263059482460946
 46808832942615408698 x_5^{95} -
 22597244646886550293078056370393329489404383276336677456171260270575647121965608337549367316252638863990999613113150114620
 72671019300635050048 x_5^{94} +
 28031880378023975171918667387500994519280618075336086932158347006240582466329666264537234857543990217318762234697012190043
 18019354780247752102 x_5^{93} -
 2908925550462696892436180046253875728763309778777324974718727706835499245661123896717969138624170884825368135175763189069
 73135074449468395158 x_5^{92} +
 26645021033609399461262534085181854856939338699023228792812734128507042303568792307871574093207225493150390219840028345114
 74969500545023873550 x_5^{91} -
 21814922959691964094476059682501210954703338243601457652314421767688853540495125406400582874348545913416657521694300262271
 13470519156640604723 x_5^{90} +
 157586548371358891834606806192411213457185294988694667242400885689975561363363561580029598485923625345885310278805771313
 29351010801960397708 x_5^{89} -
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SCHOOL OF MATHEMATICAL SCIENCES AND LPMC, NANKAI UNIVERSITY, TIANJIN 300071, P.R. CHINA
E-mail address: `chenhuibin@mail.nankai.edu.cn`, `chenzhiqi@nankai.edu.cn`, `dengsq@nankai.edu.cn`